Introduction To Two Dimensional Fractional Fourier-Mellin Transform

V. D. Sharma[1],
Department of Mathematics,
Arts, Commerce and Science College,
Kiran Nagar,Amravati, (India)

P. B. Deshmukh[2]
Department of Mathematics,
Dr. Rajendra Gode Institute of Technology & Research,
Amravati, (India)

Abstract-

The methods of integral transforms are very efficient to solve and research, differential and integral equations of mathematical physics and also studying the behavior of linear systems. With the help of these transforms many problems of the oscillation theory, heat conductivity, neutron diffusion and slowing-down, hydrodynamics, the elasticity theory, and physical kinetics can be solved.

In this paper two dimensional fractional Fourier-Mellin transform is extended in the distributional generalized sense. Analyticity of the distributional generalized two dimensional fractional Fourier-Mellin transform is proved.

Keywords: Fourier- Two-Dimensional Fractional Fourier-Mellin Transform, Testing Function Space,
Generalized function

I. Introduction:

Integral transform have been successfully used for almost two centuries in solving many problems in applied mathematics, mathematical physics and engineering science. Fourier transform diagonalizes all linear time-invariant operators, which are building blocks of signal processing and many other branches.Application of fractional Fourier transform have been reported in the solution of differential equations, optical beam propogation, signal detection, correlation etc. The Mellin transformation is a basic tool for analyzing the behavior of many important functions in mathematics and mathematical physics, such as the zeta functions occurring in number theory and in connection with various spectral problems [2].

The first investigators to use a version of the Fourier-Mellin transform were Brousil and Smith [1967][5].The Fourier-Mellin moments were firstly proposed by Sheng and Duvernoy in 1986, it was defined in a polar coordinate system [1]. Due to shift invariant property of Fourier transform, the Fourier-Mellin transform (FMT) is a very powerful tool in image restoration, pattern recognition [5]. The image reconstruction from a finite set of orthogonal Fourier-Mellin moments is examined [2]. An invariant image descriptors based on the circular-Fourier radial-Mellin transform is called the Fourier-Mellin descriptors. Yunlong Sheng and Henri H. Arsenault accomplished an invariant multiclass pattern recognition using Fourier-Mellin descriptors [3]. J. R. Martinez-de Dios and A. Ollero discussed about a robust real-time image stabilization system based on the Fourier-Mellin transform. This system is capable of performing image capture-stabilization-display at a rate of standard video on a general Pentium III at 800 MHz without any specialized hardware and the use of any particular software platforms [4].Watermarking method for the protection of multimedia signals (image, sound) in which Fourier-Mellin transform is used as a tool was described by Kin etal [10].

Motivated by the above work, we have generalized two dimensional fractional Fourier-Mellin transform in the distributional sense. In the present work an introduction to two dimensional fractional Fourier-Mellin transform is given and its Analyticity is proved. The notations and terminologies are as per Zemanian [7,8].

II. Two-dimensional fractional Fourier-Mellin transform

Definition of two-dimensional fractional Fourier-Mellin transform:

The two-dimensional fractional Fourier-Mellin transform with parameters α and θ of $f(x,y,t,q)$ denoted by 2DFRFMT{ $f(x,y,t,q)$} performs a linear operation, given by the integral transform.

2DFRFMT

$\{ f(x,y,t,q)\} = F_{\alpha,\theta}(\xi,\eta,\lambda,\chi) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} f(x,y,t,q)K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)dxdydtdq$----(2.1)

where $K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi) = \sqrt{\frac{1-icot\alpha}{2\pi}}\, e^{\frac{1}{2sin\alpha}[(x^2+y^2+\xi^2+\eta^2)cos\alpha-2(x\xi+y\eta)}\, t^{\frac{2\pi i\lambda}{sin\theta}-1}\, q^{\frac{2\pi i\chi}{sin\theta}-1}\, e^{\frac{\pi i}{tan\theta}[\lambda^2+\chi^2+log^2t+log^2q]}$

$= C_{1\alpha}e^{iC_{2\alpha}[(x^2+y^2+\xi^2+\eta^2)cos\alpha-2(x\xi+y\eta)}\, t^{C_{1\theta}i\lambda-1}\, q^{C_{1\theta}i\chi-1}\, e^{C_{1\theta}i[\lambda^2+\chi^2+log^2t+log^2q]}$

where $C_{1\alpha} = \sqrt{\frac{1-icot\alpha}{2\pi}}$, $C_{2\alpha} = \frac{1}{2sin\alpha}\, C_{1\theta} = \frac{2\pi}{sin\theta}$, $C_{2\theta} = \frac{\pi}{tan\theta}$ $\quad 0<\alpha<\frac{\pi}{2},\ 0<\theta<\frac{\pi}{2}.$ ---(2.2)

The Test Function

An infinitely differentiable complex valued smooth function $\emptyset(x,y,t,q)$ on R^n belongs to $E(R^n)$, if for each compact set $I \subset S_{a,b}$, $J \subset S_{c,d}$ where

$S_{a,b} = \{x,y: x,y\epsilon R^n, |x| \le a, |y| \le b, a>0, b>0\}$

$S_{c,d} = \{t,q: t,q\epsilon R^n, |t| \le c, |q| \le d, c>0, d>0\}$

$\gamma_{E,m,n,k,l}[\emptyset(x,y,t,q)] = \sup_{\substack{x,y\epsilon I\\ t,q\epsilon J}} \left| D^{m,n,k,l}_{x,y,t,q}\emptyset(x,y,t,q)\right| < \infty$---(2.3)

Thus $E(R^n)$ will denote the space of all $\emptyset(x,y,t,q)\epsilon E(R^n)$ with compact support contained in $S_{a,b} \cap S_{c,d}$.

Note that the space E is complete and therefore a Frechet space. Moreover, we say that $f(x,y,t,q)$ is a fractional Fourier-Mellin transformable if it is a member of E.

III. Distributional Two Dimensional Fractional Fourier-Mellin Transform (2DFRFMT)

The two dimensional distributional Fractional Fourier -Mellin transform of $f(x,y,t,q)\epsilon E^*(R^n)$ can be defined by

$2DFRFMT\{f(x,y,t,q)\} = F_{\alpha,\theta}(\xi,\eta,\lambda,\chi) = \langle f(x,y,t,q), K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)\rangle$---(3.1)

where, $K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi) = \sqrt{\frac{1-icot\alpha}{2\pi}}\, e^{\frac{1}{2sin\alpha}[(x^2+y^2+\xi^2+\eta^2)cos\alpha-2(x\xi+y\eta)}\, t^{\frac{2\pi i\lambda}{sin\theta}-1}\, q^{\frac{2\pi i\chi}{sin\theta}-1}\, e^{\frac{\pi i}{tan\theta}[\lambda^2+\chi^2+log^2t+log^2q]}$

$= C_{1\alpha}e^{iC_{2\alpha}[(x^2+y^2+\xi^2+\eta^2)cos\alpha-2(x\xi+y\eta)}\, t^{C_{1\theta}i\lambda-1}\, q^{C_{1\theta}i\chi-1}\, e^{C_{1\theta}i[\lambda^2+\chi^2+log^2t+log^2q]}$

where $C_{1\alpha} = \sqrt{\frac{1-icot\alpha}{2\pi}}$, $C_{2\alpha} = \frac{1}{2sin\alpha}, C_{1\theta} = \frac{2\pi}{sin\theta}$, $C_{2\theta} = \frac{\pi}{tan\theta}$ $\quad 0<\alpha<\frac{\pi}{2},\ 0<\theta<\frac{\pi}{2}.$ ---(3.2)

Right hand side of equation (3.1) has a meaning as the application of $f(x,y,t,q)\epsilon E^*(R^n)$to

$K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)\epsilon E$.

It can be extended to the complex space as an entire function given by

$2DFRFMT\{f(x,y,t,q)\} = F_{\alpha,\theta}(\xi',\eta',\lambda',\chi')$

$= \langle f(x,y,t,q), K_{\alpha,\theta}(x,y,t,q,\xi',\eta',\lambda',\chi')\rangle$---(3.3)

The right hand side is meaningful because for each $\xi',\eta',\lambda',\chi'\epsilon C^n$, $K_{\alpha,\theta}(x,y,t,q,\xi',\eta',\lambda',\chi')\epsilon E$ as a function of x,y,t,q.

IV. Analyticity Theorem-

Statement- let, $f(x,y,t,q)\epsilon E(R^n)$ and let its fractional Fourier-Mellin transform be defined by $2DFRFMT\{f(x,y,t,q)\} = F_{\alpha,\theta}(\xi,\eta,\lambda,\chi)$

$$= \langle f(x,y,t,q), K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)\rangle$$

Then $F_{\alpha,\theta}(\xi,\eta,\lambda,\chi)$ is analytic on C^n if the $Supp\ f \subset S_{a,b} \cap S_{c,d}$

where, $S_{a,b} = \{x,y: x,y\epsilon R^n, |x| \leq a, |y| \leq b, a > 0, b > 0\}$

$$S_{c,d} = \{t,q: t,q\epsilon R^n, |t| \leq c, |q| \leq d, c > 0, d > 0\}$$

Moreover, $F_{\alpha,\theta}(\xi,\eta,\lambda,\chi)$ is differentiable and

$D_{\xi,\eta}^{m,n}\{F_{\alpha,\theta}(\xi,\eta,\lambda,\chi)\} = \langle f(x,y,t,q), D_{\xi,\eta}^{m,n}\{K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)\}\rangle$---------------(4.1)

Proof:- let, $\xi: (\xi_1,\xi_2,---\xi_p,---\xi_m)\epsilon C^n$

$$\eta: (\eta_1,\eta_2,---\eta_q,---\eta_n)\epsilon C^n$$

We first prove that

$$\frac{\partial}{\partial\xi_p}\{F_{\alpha,\theta}(\xi,\eta,\lambda,\chi)\} = \langle f(x,y,t,q), \frac{\partial}{\partial\xi_p}\{K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)\}\rangle$$

For fixed $\xi_p \neq 0$, choose two concentric circles C_1 and C_2 with centre at ξ_p with radii r_1 and r_2 respectively such that $0 < r_1 < r_2 < |\xi_p|$, let $\Delta\xi_p$ be a complete increment satisfying $0 < |\Delta\xi_p| < r_1$.

Consider,

$$\frac{F_{\alpha,\theta}(\xi_p + \Delta\xi_p,\eta,\lambda,\chi) - F_{\alpha,\theta}(\xi,\eta,\lambda,\chi)}{\Delta\xi_p} - \langle f(x,y,t,q), \frac{\partial}{\partial\xi_p}\{K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)\}\rangle$$

$= \langle f(x,y,t,q), \psi_{\Delta\xi_p}(x,y,t,q)\rangle$-------------(4.2)

$= \langle f(x,y,t,q), \psi_{\Delta\xi_p}(x,y,t,q)\rangle$

where, $\psi_{\Delta\xi_p}(x,y,t,q) = \frac{1}{\Delta\xi_p}\{K_{\alpha,\theta}(x,y,t,q,\xi_p + \Delta\xi_p,\eta,\lambda,\chi) - K_{\alpha,\theta}(x,y,t,q,\xi_p,\eta,\lambda,\chi)\}$

$-\frac{\partial}{\partial\xi_p}\{K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)\}$----------------(4.3)

for any fixed $x,y\epsilon R^n$ and fixed integer $m = m_1, m_2, ----m_n$

$D_x^m\{K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)\}$

$= D_x^m\{C_{1\alpha}\beta_1(y)e^{iC_{2\alpha}[(x^2+\xi^2)cos\alpha - 2x\xi]}\beta_2(t,q)\}$

where, $C_{1\alpha} = \sqrt{\frac{1-icot\alpha}{2\pi}}$, $C_{2\alpha} = \frac{1}{2sin\alpha} C_{1\theta} = \frac{2\pi}{sin\theta}$, $C_{2\theta} = \frac{\pi}{tan\theta}$

$\beta_1(y) = e^{iC_{2\alpha}[(y^2+\eta^2)cos\alpha - 2y\eta}$

$\beta_2(t,q) = t^{C_{1\theta}i\lambda-1} q^{C_{1\theta}i\chi-1} e^{C_{1\theta}i[\lambda^2+\chi^2+log^2 t+log^2 q]}$

$= C_{1\alpha}\beta_1(y)\sum_{r=0}^{m} C_m\, C_{\alpha'}(xcos\alpha - \xi)^{m-2r} e^{(m-r)u}\beta_2(t,q)$

where,

$$C_m = \frac{m!}{r!\,(m-2r)!}(2i)^{m-r}$$

$$C_{\alpha'} = (cos\alpha)^r (C_{2\alpha})^{m-r}$$

$$u = iC_{2\alpha}[(x^2+\xi^2)cos\alpha - 2x\xi]$$

Since for any fixed $x\epsilon R^n$, fixed m &$0 < \alpha \le \frac{\pi}{2}$, $D_x^m\{K_{\alpha,\theta}(x,y,t,q,\tilde{\xi},\eta,\lambda,\chi)\}$ is analytic inside and on C', we have by Cauchy integral formula

$$D_x^m\psi_{\Delta\xi_p}(x,y,t,q) = \frac{1}{2\pi i}\int_{C_1} D_x^m\{K_{\alpha,\theta}(x,y,t,q,\tilde{\xi},\eta,\lambda,\chi)\}\left[\frac{1}{\Delta\xi_p}\left(\frac{1}{z-\xi_p-\Delta\xi_p} - \frac{1}{z-\xi_p}\right) - \left(\frac{1}{(z-\xi_p)^2}\right)\right] dz$$

where, $\tilde{\xi} = \xi_1, \xi_2, ---\xi_{p-1}, z, \xi_{p+1} --- \xi_n$

$$= \frac{\Delta\xi_p}{2\pi i}\int_{C_1}\frac{A(x,y,t,q,\tilde{\xi},\eta,\lambda,\chi)}{(z-\xi_p-\Delta\xi_p)(z-\xi_p)^2} dz$$

But for all $z\epsilon C_1$ and x is restricted to compact subset of R^n, $0 < \alpha \le \frac{\pi}{2}$

$A(x,y,t,q,\tilde{\xi},\eta,\lambda,\chi) = D_x^m K_{\alpha,\theta}(x,y,t,q,\tilde{\xi},\eta,\lambda,\chi)$is bounded by a constant Q.

Moreover, $\left|z-\xi_p-\Delta\xi_p\right| > r_1 - r > 0$&$\left|z-\xi_p\right| = r_1$

We have

$$\left|D_x^m\psi_{\Delta\xi_p}(x,y,t,q)\right| = \left|\frac{\Delta\xi_p}{2\pi i}\int_{C_1}\frac{A(x,y,t,q,\tilde{\xi},\eta,\lambda,\chi)}{(z-\xi_p-\Delta\xi_p)(z-\xi_p)^2} dz\right|$$

$$\le \frac{\left|\Delta\xi_p\right|}{2\pi}\int_{C_1}\frac{Q}{(r_1-r)r_1^2}|dz|$$

$$\le \frac{\left|\Delta\xi_p\right| Q}{(r_1-r)r_1}$$

Similarly,

$$\left|D_y^n\psi_{\Delta\eta_q}(x,y,t,q)\right| \le \frac{\left|\Delta\eta_q\right| P}{(r_1-r)r_1}$$

where,

$B(x,y,t,q,\xi,\tilde{\eta},\lambda,\chi) = D_y^n\psi_{\Delta\eta_q}(x,y,t,q)$is bounded by a constant P.

Thus as $\left|\Delta\xi_p\right| \to 0$. $D_x^m\psi_{\Delta\xi_p}(x,y,t,q)$ tends to zero uniformly on the compact subset of R^n, therefore it follows that $\psi_{\Delta\xi_p}(x,y,t,q)$ converges in $E(R^n)$ to zero.

∴ $F_{\alpha,\theta}(\xi,\eta,\lambda,\chi)$is differentiable with respect to ξ_p, η_q.

But this is true for all $p = 1,2,---$, $q = 1,2,---$

Hence $F_{\alpha,\theta}(\xi,\eta,\lambda,\chi)$ is analytic on C^n.

$D_{\xi,\eta}^{m,n}\{F_{\alpha,\theta}(\xi,\eta,\lambda,\chi)\} = \langle f(x,y,t,q), D_{\xi,\eta}^{m,n}\{K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)\}\rangle$

Hence proved.

V. Conclusion-

In this paper two-dimensional fractional Fourier-Mellin transform is generalized in the distributional sense. Analyticity theorem for two-dimensional fractional Fourier-Mellin transform is proved.

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