Intersection of an Ellipsoid and a Plane

Sebahattin Bektas

Ondokuz Mayis University,

Faculty of Engineering,

Geomatics Engineering, Samsun

Abstract

The intersection topic is quite popular at an interdisciplinary level. It can be the friends of geometry, geodesy, satellite orbits in space, all sorts of elliptical motions (e.g., planetary motions), curvature of surfaces and concerning eye-related radio-therapy treatment, for example the anterior surface of the cornea is often represented as ellipsoidal in form.

We have developed an algorithm for intersection of an ellipsoid and a plane with a closed form solution. To do this, we rotate the ellipsoid and the plane until inclined plane moves parallel to the XY plane. In this situation, the intersection ellipse and its projection will be the same.

This study aims to show how to obtain the center, the semi-axis and orientation of the intersection ellipse.

Keywords: ellipsoid, normal section curve, 3D reverse transformation, intersection ellipsoid and a plane

Introduction

The general ellipsoid is treated in detail as a special case. We will focus here on general ellipsoid from quadratic surfaces. Because the ellipsoid is a general surface, the ellipsoidal formulas can be used easily for rotating ellipsoid and sphere Bektas (2014), Bektas (2015-a).

This intersection issue is very important in geodesy. To make geodetic computations on the ellipsoid (rotational or triaxial) first we need to know the normal section curve that combines observation points. The normal section curve is also available from the intersection of the ellipsoid and a plane which contains normal of surface on the station point and passes from destination point. This current study aims to pave the way for our further study on triaxial ellipsoid work. Today, basic invers and forward problem between the two points on the triaxial ellipsoid with geodetic coordinates could not be a clear solution. Our future work will be on this unsolvable problem. I hope, the result of this study will contribute to the solution of the above problem.

Basically everyone knows that intersection of a sphere and plane is a circle. But when we get common solution the plane and the sphere equation that will give us an ellipse not a circle. While the solution in R3 space we have to eliminate one of the X,Y and Z parameters. In this case it is clear that there will be three possible solutions. Three of them are generally different ellipses from each other. For example, if we eliminate the parameter Z, we get the following ellipse equation on the XY plane that is the projection of the true intersection circle.

Similarly we find the intersection an ellipsoid and a plane is an ellipse with a common solution. But it is not true intersection ellipse. That is projection of the true intersection ellipse. The intersecting ellipse's plane is not parallel to the XY plane. This is why the two ellipses are different from each other.

As related to this subject limited number of studies was found in literature. Some of them are Klein (2012), Ferguson (1979), URL1, URL2,URL3. I think Klein's study is a good study. But understanding his study requires familiarity with differential geometry. In this study we have put forward an alternative method addition to the Klein's study. We believe that our method is easier than to understand Klein's.

Our method is an easy way to understand the unfamiliar differential geometry. As also differently, we calculate the intersection ellipse orientation information. Because the orientation information is extremely necessary especially in the curvature of surface.

Here, our aim is to achieve the true intersection ellipse. To do this, we rotate the ellipsoid and the plane until inclined plane moves parallel to the XY plane. In this situation, the intersection ellipse and its projection will be the same. Of course, in this case we will need to use the new equation of ellipsoid because the ellipsoid is no longer in standard position, it is rotated and shifted. The same situation is also valid for the intersection of plane and rotational ellipsoid, hyperboloid and other quadratic surfaces.

Definition

Ellipsoid

An ellipsoid is a closed quadric surface that is analogue of an ellipse. Ellipsoid has three different axes (a>b>c) as shown in Fig.1. Mathematical literature often uses "ellipsoid" in place of "Triaxial ellipsoid or general ellipsoid". Scientific literature (particularly geodesy) often uses "ellipsoid" in place of "biaxial ellipsoid, rotational ellipsoid or revolution ellipsoid". Previous literature uses 'spheroid' in place of rotational ellipsoid. The standard equation of an ellipsoid centered at the origin of a Cartesian coordinate system and aligned with the axes as indicated in Eq.1.

Although ellipsoid equation is quite simple and smooth, computations are quite difficult on the ellipsoid. The main reason for this difficulty is the lack of symmetry. Generally, an ellipsoid is defined with 9 parameters. These parameters are; 3 coordinates of center (x_o, y_o, z_o), 3 semi-axes (a,b,c) and 3 rotational angles (ε, ψ, ω) which represent rotations around x-,y- and z- axes respectively. These angles control the orientation of the ellipsoid.

1.2 Intersection of an Elipsoid and a Plane

$$\frac{X^2}{a^2}+\frac{Y^2}{b^2}+\frac{Z^2}{c^2}-1=0 \qquad \text{(Ellipsoid equation)} \qquad (1)$$

$$A_X X + A_Y Y + A_Z Z + A_D = 0 \qquad \text{(Plane equation)} \qquad (2)$$

Let's assume that X_0, Y_0, Z_0 are the coordinates of the center of the intersection ellipse and a_i, b_i are the major and minor semi-axes of the intersection ellipse
We can start with the common solution of two equations (Eq.1-2). If we eliminate the parameter Z , we get the following ellipse equation on the XY plane that is the projection of the intersection ellipse.

$$A X^2 + B X Y + C Y^2 + D X + E Y + F = 0 \qquad (3)$$

These coefficients are calculated from the common solution is obtained from the two equations the ellipsoid and plane equation

$$A = 1/a^2 + (A_x^2) / (A_z^2 c^2)$$
$$B = (2 A_x A_y) / (A_z^2 c^2)$$
$$C = 1/b^2 + (A_y^2) / (A_z^2 c^2) \qquad (4)$$
$$D = (2 A_x A_D) / (A_z^2 c^2)$$
$$E = (2 A_y A_D) / (A_z^2 c^2)$$
$$F = A_D^2 / (A_z^2 c^2) - 1$$

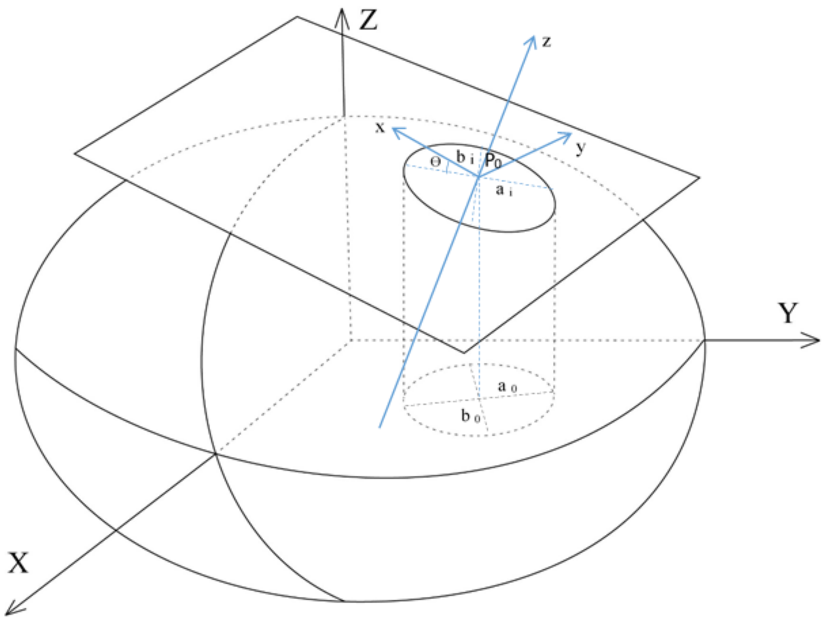


Fig.1 Intersection ellipsoid and plane

When we solve the Eq.3, we get five ellipse parameters. They are:

X_0, Y_0 (center of ellipse in XY plane)

a_o , b_o (major and minor semi axis of ellipse in XY plane)

θ (orientation angle between X axis and semi major axis)

$$M_o = \begin{bmatrix} F & D/2 & E/2 \\ D/2 & A & B/2 \\ E/2 & B/2 & C \end{bmatrix} \qquad M = \begin{bmatrix} A & B/2 \\ B/2 & C \end{bmatrix} \tag{5}$$

λ_1, λ_2 : eigenvalues of M matrices ($\lambda_1 < \lambda_2$)

$$a_o = \sqrt{-\det(M_o)/(\det(M)\lambda_1)} \quad \text{(major semi-axis of intersection ellipse)} \tag{6}$$

$$b_o = \sqrt{-\det(M_o)/(\det(M)\lambda_2)} \quad \text{(minor semi-axis of intersection ellipse)} \tag{7}$$

$$X_0=(BE-2CD)/(4AC-B^2) \tag{8}$$

$$Y_0=(BD-2AE)/(4AC-B^2) \quad \text{(coordinates of intersection ellipse's center)} \tag{9}$$

$$Z_0=-(A_x X_o + A_y Y_o + A_D)/A_z \tag{10}$$

$$\tan 2\theta = \frac{B}{A-C} \quad \text{(orientation angle of projection ellipse)} \tag{11}$$

Now we rotate together the ellipsoid and the plane (fig.1) until inclined plane move parallel to the XY plane. In this situation the intersection ellipse and its projection will be the same.

For this the origin of the XYZ system must be moved to points of P_o (X_0 ,Y_0, Z_0). We need the transformation parameters.

R_{3x3}-rotation matrix is obtained from the rotational angles

$$R = \begin{bmatrix} \cos\psi\cos\omega & \cos\varepsilon\sin\omega + \sin\varepsilon\sin\psi\cos\omega & \sin\varepsilon\sin\omega - \cos\varepsilon\sin\psi\cos\omega \\ -\cos\psi\sin\omega & \cos\varepsilon\cos\omega - \sin\varepsilon\sin\psi\sin\omega & \sin\varepsilon\cos\omega + \cos\varepsilon\sin\psi\sin\omega \\ \sin\psi & -\sin\varepsilon\cos\psi & \cos\varepsilon\cos\psi \end{bmatrix} \tag{12}$$

This is a 3D transformation equation without scale.

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} X_o \\ Y_o \\ Z_o \end{bmatrix} + R.\begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad (13)$$

This transformation equation can be written more simply with **T** Expanded transformation matrix as follows Bektas (2015-b).

T_{4x4}- expanded transformation matrix is obtained from the R_{3x3} rotational matrix and the shifted parameters (X_0, Y_0, Z_0)

$$\mathbf{T} = \begin{bmatrix} & R_{3x3} & & X_0 \\ & & & Y_0 \\ & & & Z_0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \mathbf{T^{-1}} = \begin{bmatrix} & R^T_{3x3} & & X_{0T} \\ & & & Y_{0T} \\ & & & Z_{0T} \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (14)$$

$$\begin{bmatrix} X_i \\ Y_i \\ Z_i \\ 1 \end{bmatrix} = \mathbf{T} \begin{bmatrix} x_i \\ y_i \\ z_i \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} x_i \\ y_i \\ z_i \\ 1 \end{bmatrix} = \mathbf{T^{-1}} \begin{bmatrix} X_i \\ Y_i \\ Z_i \\ 1 \end{bmatrix} \quad (15)$$

Determination of transformation parameters

Shifted parameter X_0 ,Y_0, Z_0 are intersection of ellipse's center coordinates that is founded before (Eq.8-10). We must find three rotation angles (ε, Ψ, ω). For this, we take advantage of the nearest plane's point from the origin. The point Q on a plane $A_X.X + A_Y.Y + A_Z.Z + A_D = 0$ that is closest to the origin has the Cartesian coordinates (q_x, q_y, q_z) Shifrin and Adams (2010).

where

$$q_x = \frac{A_x A_D}{\sqrt{A_x^2 + A_y^2 + A_z^2}} \qquad q_y = \frac{A_y A_D}{\sqrt{A_x^2 + A_y^2 + A_z^2}} \qquad q_z = \frac{A_z A_D}{\sqrt{A_x^2 + A_y^2 + A_z^2}} \quad (16)$$

And rotation angles (ε, Ψ, ω)

□□□□□ (17)

$$\psi = \pi/2 - \arctan\left(\frac{|q_z|}{\sqrt{q_x^2 + q_y^2}}\right) \quad (18)$$

$$\omega = \pi - \arctan\left(\frac{q_y}{q_x}\right) \quad (19)$$

Of course, in this case we will need to use the new equation of ellipsoid. Because the ellipsoid is no longer in standard position it is rotated and shifted. We have to reverse 3D transformation to the new ellipsoid parameters.

Before we found transformation parameter X_0 ,Y_0, Z_0 , ε, Ψ, ω . These parameters are used transformation from XYZ to xyz. Let's see how to find the reverse transform parameters (X_{0T} ,Y_{0T}, Z_{0T} , ε_T, Ψ_T, ω_T) for the transformation from xyz to XYZ

To find inverse transformation parameters we can take advantage of the inverse of the T matrix The reverse transformation parameters(X_{0T} ,Y_{0T}, Z_{0T} , ε_T, Ψ_T, ω_T) are located to T^{-1} inverse matrix. Reverse shifted parameters (X_{0T} ,Y_{0T}, Z_{0T}) is located the inverse matrix T^{-1} in column fourth. Reverse rotation angles are calculated from the elements of the matrix R as follows

$$\varepsilon_T = -\arctan(R_{23}/R_{33}) \quad (20)$$
$$\Psi_T = \arcsin(R_{13}) \quad (21)$$
$$\omega_T = -\arctan(R_{12}/R_{11}) \quad (22)$$

Now we can write a new ellipsoid equation rotated and shifted, to do this, we put (Eq.13). into (Eq.1) standard ellipsoid equation

$$1/a^2 [(X-X_{0T}) \cos\Psi_T \cos\omega_T +(Y-Y_{0T})(-\cos\Psi_T) \sin\omega_T +(Z-Z_{0T}) \sin\Psi_T]^2$$
$$+ 1/b^2 [(X-X_{0T})(\cos\varepsilon_T \sin\omega_T + \sin\varepsilon_T \sin\Psi_T \cos\omega_T)$$
$$+ (Y-Y_{0T}) (\cos\varepsilon_T \cos\omega_T - \sin\varepsilon_T \sin\Psi_T \sin\omega_T) - (Z-Z_{0T}) \sin\varepsilon_T) \cos\Psi_T]^2$$
$$+ 1/c^2[(X-X_{0T})(\sin\varepsilon_T \sin\omega_T - \cos\varepsilon_T \sin\Psi_T \cos\omega_T)$$
$$+ (Y-Y_{0T})(\sin\varepsilon_T \cos\omega_T +\cos\varepsilon_T \sin\Psi_T \sin\omega_T)+ (Z-Z_{0T}) \cos\varepsilon_T \cos\Psi_T]^2-1=0 \quad (23)$$

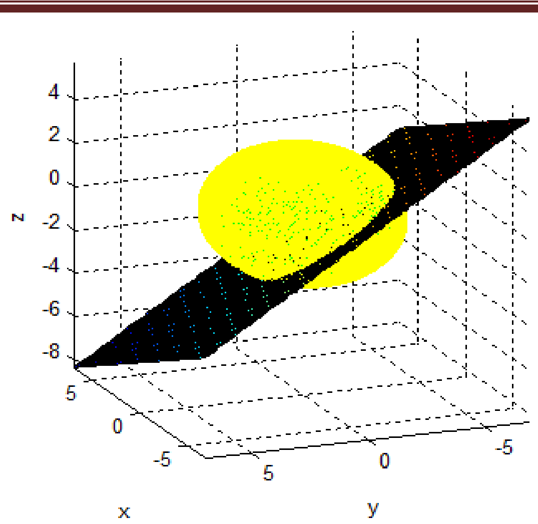
In this equation if we put z = 0 we obtain a conical intersection ellipse equation form as follows.

$$1/a^2 [(X-X_{0T}) \cos\Psi_T \cos\omega_T +(Y-Y_{0T})(-\cos\Psi_T) \sin\omega_T -Z_{0T} \sin\Psi_T]^2$$
$$+ 1/b^2 [(X-X_{0T})(\cos\varepsilon_T \sin\omega_T + \sin\varepsilon_T \sin\Psi_T \cos\omega_T)$$
$$+ (Y-Y_{0T}) (\cos\varepsilon_T \cos\omega_T - \sin\varepsilon_T \sin\Psi_T \sin\omega_T)+Z_{0T} \sin\varepsilon_T) \cos\Psi_T]^2$$
$$+ 1/c^2[(X-X_{0T})(\sin\varepsilon_T \sin\omega_T - \cos\varepsilon_T \sin\Psi_T \cos\omega_T)$$
$$+ (Y-Y_{0T})(\sin\varepsilon_T \cos\omega_T +\cos\varepsilon_T \sin\Psi_T \sin\omega_T)-Z_{0T} \cos\varepsilon_T \cos\Psi_T]^2-1=0 \quad (24)$$

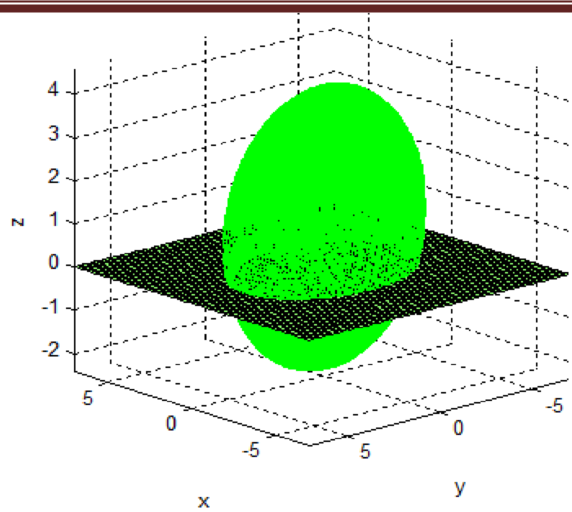
Above conic equation rearranged beloved the intersection ellipse's conic equation obtained.

$$A X^2 + B X Y + C Y^2 + D X + E Y + F=0 \quad (25)$$

When we solve the above ellipse equation, we get five ellipse parameters of intersection ellipse (X_0, Y_0 , a_i, b_i , θ) .



a) Before rotating



b) After rotating

Fig.2 Rotating together ellipsoid and plane

As a result we have presented computational results that were realized in MATLAB. Whoever wants to deal with this subject can get in contact with me and use this link for free (URL3).

1.2 Intersection of a Sphere and Plane

$X^2 + Y^2 + Z^2 - R^2 = 0$ (Sphere equation) (26)

$A_X X + A_Y Y + A_Z Z + A_D = 0$ (Plane equation) (27)

Practical solutions can be reached as follows

Firstly, we find the nearest Q point of plane to the origin. The coordinates of Q point (q_x, q_y, q_z) can be found from the given plane equation coefficients. These coordinates were founded previously (Eq.16)

The radius of the circle intersections (r)

$$r = \sqrt{R^2 - (q_x^2 + q_y^2 + q_z^2)} \quad (28)$$

Numerical Example-1 : intersection of ellipsoid and plane

$$\frac{x^2}{25} + \frac{y^2}{16} + \frac{z^2}{9} - 1 = 0$$ (Ellipsoid equation)

$X + 2Y + 3Z + 4 = 0$ (Plane equation)

Find the intersection of ellipsoid and plane given above.

Firstly, we find the equation of conic of projection ellipse in the XY plane.

$A X^2 + B Y^2 + C X Y + D X + E Y + F=0$

These coefficients are calculated from the ellipsoid and the plane equation. (Eq.4)

A= 0.05234567
B= 0.04938271
C= 0.11188271
D= 0.09876543
E= 0.19753086
F= -0.80246913

Five ellipse parameters (x_o, y_o, a_o, b_o, θ) can be calculated from (Eqs.6-11)

$X_0 = -0.5882 \quad Y_0 = -0.7529 \quad Z_0 = -0.6353$

$a_o = 4.5666 \quad b_o = 2.7385 \quad \theta = 2.795373$

The closest point Q to the origin has the Cartesian coordinates (q_x, q_y, q_z), (Eq.16)

$$q_x = \frac{A_x A_D}{\sqrt{A_x^2 + A_y^2 + A_z^2}} = -0.2857 \qquad q_y = \frac{A_y A_D}{\sqrt{A_x^2 + A_y^2 + A_z^2}} = -0.5714$$

$$q_z = \frac{A_z A_D}{\sqrt{A_x^2 + A_y^2 + A_z^2}} = -0.8571$$

And rotation angles (ε, Ψ, ω) (Eqs.17-19).

$$\psi = \pi/2 - \arctan\left(\frac{|q_z|}{\sqrt{q_x^2 + q_y^2}}\right) = 36.6992$$

$$\omega = \pi - \arctan\left(\frac{q_y}{q_x}\right) = 116.5650$$

$$R = \begin{bmatrix} -0.3586 & 0.8944 & 0.2673 \\ -0.7171 & -0.4472 & 0.5345 \\ 0.5976 & 0.0000 & 0.8018 \end{bmatrix}$$

Rotation matrix (Eq.12)

$$T=\begin{bmatrix} -0.3586 & 0.8944 & 0.2673 & -0.5882 \\ -0.7171 & -0.4472 & 0.5345 & -0.7529 \\ 0.5976 & 0.0000 & 0.8018 & -0.6353 \\ 0 & 0 & 0 & 1.0000 \end{bmatrix}$$ Expanded transformation matrix (Eq.14)

$$T^{-1}=\begin{bmatrix} -0.3586 & -0.7171 & 0.5976 & -0.3712 \\ 0.8944 & -0.4472 & 0.0000 & 0.1894 \\ 0.2672 & 0.5346 & 0.8018 & 1.0690 \\ 0.0000 & 0.0000 & 0.0000 & 1.0000 \end{bmatrix}$$ Inverse matrix (Eq.14)

Reverse translation parameters (X_{OT} ,Y_{OT}, Z_{OT}) are located to the inverse matrix T^{-1} in column 4

$X_{OT} = -0.3712$, $Y_{OT} = 0.1894$, $Z_{OT} = 1.069$

Reverse rotation angles are found from R rotation matrix (Eq.20-22)

ε_T = -arc tan (R_{23}/R_{33}) = -33.6901^o

Ψ_T = arc sin (R_{13}) = 15.5014^o

ω_T = -arc tan (R_{12}/R_{11}) = -111.8454^o

The intersection ellipse in conic form as follows. (Eq.24)

$0.076968\, X^2 + 0.014431\, X\, Y + 0.0445\, Y^2 + 0\, X + 0\, Y - 0.905882=0$

When we solve the above ellipse equation, we get five ellipse parameters of intersection ellipse. (Eqs.6-11)

$X_0= 0$ $Y_0=0$ $a_i = 4.5915$ $b_i = 3.3970$ $\theta=1.7799 rad$

Numerical Example-2: intersection of sphere and plane

$X^2 + Y^2 + Z^2 = 25$ (Sphere equation)

$X + 2Y +3Z + 4 =0$ (Plane equation)

Find the intersection of sphere and plane given above.

Firstly, we find the nearest Q point of plane to the origin. The coordinates of Q point (q_x,q_y,q_z) can be found from the given plane equation coefficients as below.

$$q_x = \frac{A_x A_D}{\sqrt{A_x^2 + A_y^2 + A_z^2}} = -0.2857 \qquad q_y = \frac{A_y A_D}{\sqrt{A_x^2 + A_y^2 + A_z^2}} = -0.5714$$

$$q_z = \frac{A_z A_D}{\sqrt{A_x^2 + A_y^2 + A_z^2}} = -0.8571$$

The radius of the circle intersections (r)

$$r = \sqrt{R^2 - (q_x^2 + q_y^2 + q_z^2)} = 4.8844$$

Conclusion

In this study, we have developed an algorithm for intersection of an ellipsoid and a plane with a closed form solution. The efficiency of the new approaches is demonstrated through a numerical example. The presented algorithm can be applied easily for spheroid, sphere and also other quadratic surface, such as paraboloid and hyperboloid.

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