Convolution Theorem for Two Dimensional Fourier-Mellin Transform

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Abstract - The convolution in one domain equals point-wise multiplication in the other domain and we know that a two-dimensional function is represented in a computer as numerical values in a matrix, whereas a one dimensional Fourier transforms in a computer is an operation on a vector. A two dimensional Fourier transforms can be computed by a sequence of one dimensional Fourier transforms. The FMT is a global transform and applies to all pixels in the same way. In the present paper, we defined the property of two dimensional Fourier–Mellin transform which is mainly concerned with the study of similarity transformations i.e. direct product of the rotation and scale groups.

In this paper, we developed the convolution theorem for the two dimensional Fourier-Mellin transform.

Index Terms - Fourier Transform, Mellin Transform, Distributional function, the testing function space, Two Dimensional Fourier-Mellin Transform.

I. INTRODUCTION

A Two-dimensional function is represented in a computer as numerical values in a matrix [2]. In the present paper, we defined the property of two dimensional Fourier-Mellin Transform i.e. we present a theorem of convolution for two dimensional Fourier-Mellin transform. If an input signal is applied to linear and time invariant system, the output can be found through convolution. The convolution in one domain equals point-wise multiplication in other domain. In the present paper, we generalized the two dimensional Fourier-Mellin transform.

We know that, the motivation for the Fourier transform comes from the study of Fourier series, since any complicated function can be written as the sum of simple waves, mathematically represented by sine and cosine functions. Fourier transform operates as the basic and the most widely used tool in no matter theoretical researches or engineering applications in the field of signal processing. Fourier analysis lies at the heart of signal processing, including audio speech, images, videos, seismic data, radio transmissions and so on. Also there are many modern technological advances e.g. television, music CD's, DVD's, cell phones, movies, computer graphics, image processing, fingerprint analysis and storage such types of many applications founded in Fourier theory[1].

Mellin transform, a kind of nonlinear transformation which is widely used for its scale invariance property. In mathematics, the Mellin transform is an integral transform that may be regarded as the multiplicative version of the two-sided Laplace transform. This integral transform is often used in number theory, mathematical statistics and the theory of asymptotic expansions; it is closely related to the Fourier transform. Due to its scale invariance property, the Mellin Transform is widely used in computer science for the analysis of algorithms. This scale invariance property is analogous to the Fourier Transform's shift invariance property; this property is useful in image recognition. The image of an object is easily scaled, when the object is moved towards or away from the camera [3]. The convolution type integral is main tool of the Mellin transform which is used to derive the filter function that convert an input noise which is difficult to be managed into desirable output noise [4].

The FMT is a global transform and applies to all pixels in the same way. The Fourier–Mellin transform is mainly concerned with the study of similarity transformations (direct product of the rotation and scale groups) [5]. Due to its resulting spectrum is invariant in rotation, translation and scale, this transform is a powerful tool for image recognition [6, 7]. Fourier-Mellin transform is used for content-based image retrieval, digital imaging, watermarking [8]. The Fourier–Mellin transform also uses the log-polar transformation to recover rotation and scale [9]. The Fourier-Mellin transform was implemented on a digital computer and also applied for the recognition and differentiation of images of plant leaves regardless of translation, rotation or scale [10]. Motivated from this work on FMT, we extent and generalized the FMT in Two dimensions.

The outline of this paper-

In this paper, we defined a two dimensional Fourier-Mellin transform in section 2. In section 3, we defined the Testing Function space. Also we defined two dimensional distributional Fourier-Mellin transform of generalized function. In section 4, we proved a theorem. Section 5 gives the definition of two dimensional Fourier-Mellin type convolutions. Section 6, 7, 8 present some theorems. The notation and terminology are given as per A.H. Zemanian [11].

II. DEFINITION of Two Dimensional Fourier-Mellin Transform

The Two dimensional Fourier-Mellin Transform with parameter of $f(t,l,x,y)$ denoted by using [12] as-

$FM\{f(t,l,x,y)\} = F(s,u,p,v) =$

$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} f(t,l,x,y)\ K(s.u.p.v)\ dt\ dl\ dx\ dy$

Where the kernel $K(s.u.p.v) = e^{-(st+ul)}\ x^{p-1} y^{v-1}$.

III. The Testing Function Space $FM_{a,b,c,d,\alpha}$

Let I be the open set in $R_+ \times R_+$ and E_+ denoted the class of infinitely differentiable function defined on I_1.

The space $FM_{a,b,c,d,\alpha}$ is given by using [13] as-

$FM_{a,b,c,d,\alpha} = \{\emptyset : \emptyset \in E_+ / \gamma_{a,b,c,d,k,r,h,g,q,j}\, \emptyset(t,l,x,y) =$
$\sup_{I_1}\left|t^k l^r \xi_{a,b}(x) x^{h+1} \eta_{c.d}(y) y^{g+1}\ D_t^q D_x^h D_l^j D_y^g \emptyset(t,l,x,y)\right|$
$< C\ A^k k^{k\alpha}\}$ ………………… (3.1)

Where, the constant C, A, B, P and T depending on testing function space $\emptyset$. The spaces $FM_{a,b,c,d,\alpha}$ are equipped with their natural Housedroff locally convex topology $T_{a,b,c,d,\alpha}$. The topology is respectively generated by the total families of semi norms $\{\gamma_{a,b,c,d,k,r,h,g,q,j}\}$ given by (3.1).

IV. Definition of Two dimensional Distributional Fourier-Mellin Transform of generalized function in $FM^*_{a,b,c,d,\alpha}$

For $f(t,l,x,y) \in FM^*_{a,b,c,d,\alpha}$, where $FM^*_{a,b,c,d,\alpha}$ is dual space of $f(t,l,x,y) \in FM_{a,b,c,d,\alpha}$ and $a < Re\ p < b,\ c < Re < d, s > 0, u > 0$.

The two dimensional Fourier-Mellin transform is defined as-

$$FM\{f(t,l,x,y)\} = F(s,u,p,v)$$
$$= \langle f(t,l,x,y), e^{-i(st+ul)} x^{p-1} y^{v-1}\rangle$$

Where, for each fixed $x(0 < x < \infty), y(0 < y < \infty$, $t0<t<\infty$, $l(0<l<\infty)$.

V. Theorem

If $f(t,l,x,y) \in FM^*_{a,b,c,d,\alpha}$ and $\emptyset(t,l,x,y) \in FM_{a,b,c,d,\alpha}$ then $\emptyset \to \psi$ is continuous linear mapping of $FM_{a,b,c,d,\alpha} \to FM_{a,b,c,d,\alpha}$, where $\psi(u,v,w,z) = \langle f(t,l,x,y), \emptyset(t+u, l+v, wx, zy)\rangle$ (5.1)

Proof: - It is easily proved that- $\emptyset(t+u, l+v, wx, zy) \in FM_{a,b,c,d,\alpha}$

Whenever, $\emptyset(t,l,x,y) \in FM_{a,b,c,d,\alpha}$. Thus, R.H.S. of (5.1) is meaningful.

By induction method we can show that-

$$D_{u,v,w,z}^{m+n+m'+n'} \psi(u,v,w,z) = \langle f(t,l,x,y), D_{u,v,w,z}^{m+n+m'+n'} \emptyset(t+u, l+v, wx, zy)\rangle$$

{Symbol $D_{u,v,w,z}^{m+n+m'+n'}$ means m[th] derivative of u, nth derivative of v, m'^{th} derivative of w and n'^{th} derivative of z}

For showing $\psi(u,v,w,z) \in FM_{a,b,c,d,\alpha}$.

Now consider,

$$\gamma_{a,b,c,d,k,r,h,g,q,j}\ \psi(u,v,w,z)$$
$$= \sup_{I_1}\left|u^k v^r \xi_{a,b}(w) w^{h+1} \eta_{c.d}(z) z^{g+1}\ D_u^q D_w^h D_v^j D_z^g \psi(u,v,w,z)\right|$$
$$= \sup_{I_1}\left|u^k v^r \xi_{a,b}(w) w^{h+1} \eta_{c.d}(z) z^{g+1}\ D_u^q D_w^h D_v^j D_z^g \langle f(t,l,x,y), \emptyset(t + u, l+v, wx, zy)\rangle\right|$$

{Since, by given term}

$$\le C \max_{\substack{0\le m\le r_1\\ 0\le n\le r_2\\ 0\le m'\le r_3\\ 0\le n'\le r_4}} \sup_{I_1}\left|u^k v^r \xi_{a,b}(w) w^{h+1} \eta_{c.d}(z) z^{g+1}\ D_u^q D_w^h D_v^j D_z^g t^k l^r \xi \ldots + u, l+v, wx, zy)\right|$$

$$\le C \max_{\substack{0\le m\le r_1\\ 0\le n\le r_2\\ 0\le m'\le r_3\\ 0\le n'\le r_4}} \sup_{I_1}\left|u^k v^r \xi_{a,b}(w) w^{h+1} \eta_{c.d}(z) z^{g+1}\ t^k l^r \xi_{a,b}(x) x^{h+1} \eta \ldots + u, l+v, wx, zy)\right|$$

$$\le C\ \max_{\substack{0\le m\le r_1\\ 0\le n\le r_2\\ 0\le m'\le r_3\\ 0\le n'\le r_4}} \gamma_{a,b,c,d,k,r,q+m,h+n,j+m',g+n'}\ \emptyset$$

(5.2)

Thus, $\psi(u,v,w,z) \in FM_{a,b,c,d,\alpha}$ continuity follows from (5.2) and hence the theorem.

VI. Definition of two dimensional Fourier-Mellin type convolutions

Let a, b, c, d are real numbers with $a \le b \le c \le d$, two dimensional Fourier-Mellin type convolution is an operation that assign to each arbitrary choice of the pair $f \in FM^*_{a,b,c,d,\alpha}$ and $g \in FM^*_{a,b,c,d,\alpha}$. The two dimensional Fourier-Mellin type convolution $f * g \in FM^*_{a,b,c,d,\alpha}$ defined by

$$\langle f * g, \emptyset\rangle = \langle f(t,l,x,y), \langle g(u,v,w,z), \emptyset(t+u, l + v, wx, zy)\rangle\rangle$$

Where $\emptyset \in FM_{a,b,c,d,\alpha}$ (6.1)

VII. Theorem

If $f \in FM^*_{a,b,c,d,\alpha}$ and $g \in \mathcal{D}_+(I)$ then $g \to f * g$ is continuous, linear mapping from $\mathcal{D}_+(I)$ into E_+, where, $(f * g)(u,v,w,z) = \langle f(t,l,x,y), \frac{1}{xy} g(u-t, v-l, wx, zy)$.

Proof:

It is easy to prove that $(f * g)$ is smooth and mapping is linear.

For its continuity-

$$\left|D^{k_1+k_2+k_3+k_4}_{u,w,v,z}(f * g)(u,v,w,z)\right| = \left|\langle f(t,l,x,y), D^{k_1+k_2+k_3+k_4}_{u,w,v,z}\left\{\frac{1}{xy} g(u-t, v-l, \frac{w}{x}, \frac{z}{y})\right\}\rangle\right|$$

$$\le C \max_{\substack{0\le q+k_1\le r_1\\ 0\le h+k_2\le r_2\\ 0\le j+k_3\le r_3\\ 0\le g+k_4\le r_4}} \sup_{I_1}\left|D^{q+h+j+g}_{t,x,l,y}\, D^{k_1+k_2+k_3+k_4}_{u,w,v,z}\left\{\frac{1}{xy} g(u-t, v-l, \frac{w}{x}, \frac{z}{y})\right\}\right|$$

Since, $g \in \mathcal{D}_+(I)$, continuity follows from the above inequality.

We have-

$(f * g)(u,v,w,z) = \langle f(t,l,x,y), \frac{1}{xy} g(u-t, v-l, wx, zy)$

as two dimensional Fourier-Mellin type.

VIII. Theorem

Convolution operation in (VI) commutes with shifting scaling operator s i.e. $s(f * g) = f * (s(g))$.

Proof: - Consider,

$$\langle s(f * g), \emptyset(t,l,x,y)\rangle = \langle (f * g), \emptyset(t+u, l+v, wx, zy)\rangle = \langle f, \langle g, \frac{1}{xy}\emptyset(u-t, v-l, wx, zy) \quad \ldots\ldots\ldots\ldots (8.1)$$

Now,

$$\langle s(f * g), \emptyset(t,l,x,y)\rangle = \langle f * (s(g)), \emptyset(t,l,x,y)\rangle$$
$$= \langle f, \langle s(g), \frac{1}{xy}\emptyset(t+u, l+v, wx, zy)$$
$$= \langle f, \langle g, s\emptyset(t+u, l+v, wx, zy)$$
$$= \langle f, \langle g, \frac{1}{xy}\emptyset(u-t, v-l, wx, zy) \ldots\ldots\ldots\ldots (8.2)$$

Theorem follows from (8.1) and (8.2).

IX. Theorem

$f \in \mathcal{D}_+(I)$ and $FM\{f(u,v,w,z)\} = F(s,u',p,v')$, s,u',p and $v' \in \Omega_f$ and if $g \in \mathcal{D}^*_+$, $FM\{g(t,l,x,y)\} = G(s,u',p,v')$. s,u',p and $v' \in \Omega_g$ and $\Omega_f \cap \Omega_g$ is not empty, then $f * g$ exist in the sense of FM- type convolution in $FM^*_{a,b,c,d,\alpha}$. Where, the strip of definition is the intersection of $\Omega_f \cap \Omega_g$ with real axis. Moreover, $FM(f * g) = FM(f) * FM(g)$.

Proof: - Using theorem (VII) it can be easily shown that-

$f * g \in FM^*_{a,b,c,d,\alpha}$.

Further as $e^{-i(st+u'l)}\, x^{p-1}\, y^{v'-1}$ for each fixed s,u',p,v' with $a < Re\, p < b,\ c < Re v' < d, s > 0, u' > 0$.

$$FM\{f * g\} = \langle f * g, e^{-i(st+u'l)}\, x^{p-1}\, y^{v'-1}\rangle$$
$$= \langle f(u,v,w,z,), \langle g, e^{-is(t+u)} e^{-iu'(l+v)}\, (wx)^{p-1}\, (zy)^{v'-1}\rangle\rangle$$
$$= \langle f(u,v,w,z,), \langle g(t,l,x,y), e^{-is(t+u)} e^{-iu'(l+v)}\, (wx)^{p-1}\, (zy)$$
$$\langle f(u,v,w,z,), e^{-isu} e^{-iu'v}\, (w)^{p-1}(z)^{v'-1}\rangle\langle g(t,l,x,y), e^{-ist}$$

$\langle f(u,v,w,z,), e^{-i(su+u'v)}(w)^{p-1}(z)^{v'-1} \rangle \langle g(t,l,x,y), e^{-i(st+u'l)}(x)^{p-1}(y)^{v'-1} \rangle FM\{f * g\} = FM(f).FM(g).$

X. Conclusion

In the present, we developed the convolution theorem for the two dimensional Fourier-Mellin Transform. The convolution theorem for the two dimensional FMT may be applicable in computer vision, image processing, Signal processing, electrical engineering, natural language processing, statistics and probability.

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