Convolution Theorem for Generalized Two Dimensional Fractional Sine Transform

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Abstract:

The applications of fractional transforms to generalized function have been done time to time and their properties have been studied by various mathematicians. Fourier transform is a very powerful tool for problems in signal processing and other applications. The Fractional Fourier Transform (FrFT) is a generalization of the ordinary Fourier transform. The ordinary Fourier transform and related techniques are of importance in various different areas like communications, signal processing and control systems. In fact, the FrFT has already found many applications in the areas of signal processing and communications. The success of FrFT in its application has promoted the development of other kinds of fractional transforms like fractional Hartley transform, fractional Hadamard transform, fractional cosine transform and fractional sine transform (FrST).

In this paper convolution theorem for generalized two dimensional fractional Sine transform is proved.

Keyword: fractional cosine transforms (FrCT), fractional sine transforms (FrST), fractional Fourier transform.

Introduction:

Fourier transform (FT) is named in the honor of Joseph Fourier (1768-1830). In the theory of Integral transform, Fourier analysis is one of the most frequently used tools in signal processing and many other scientific fields. The fractional Fourier transform was proposed by Namias and developed by Mc Brideit. Furthermore, it has been studied by many researchers and contributed. The fractional calculus have several applications in various fields of Mathematics as well as in real life situations, such as Abel's integral equation, viscoelasticity, capacitor theory, conductance of biological systems. The idea of fractional operators, fractional derivative, fractional geometry has long back history but fractional transform has been rediscovered in quantum mechanics, optics, signal processing as well as in pattern recognition. Now a days, many linear boundary value and initial value problems in applied mathematics, mathematical physics, and engineering science are effectively solved by fractional Fourier and fractional Hartley transforms.

The classical theory of local fractional calculus introduced by Kolwankar and Gangal [3] which becomes useful tool in the areas ranging from fundamental science to engineering.

Nowadays there is no doubt the use and applications of the continuous and discrete convolution operations in many branches of science. All FrCTs and FrSTs possess convolution — multiplication property which is a powerful tool for performing digital filtering in the transform domain. The convolution operation in the transform domain realized by taking an inverse transform of the product of forward transforms of two data sequences is equivalent to symmetric convolution of those symmetrically extended sequences in the spatial domain. Convolution plays a very important role in the theory of integral transform. Almeida [3] had defined convolution for fractional Fourier transform. Zayed [4] had revised the definition in order to follow the standard Convolution theorem. In our previous work we already defined following terms.

1.1. Generalized two dimensional fractional Sine transform

Two dimensional fractional Sine transform with parameter α f(x, y) denoted by $F_s^\alpha(x,y)$ perform a linear operation given by the integral transform.

$F_s^\alpha\{f(x,y)\}(u,v) = \int_0^\infty\int_0^\infty f(x,y)K_\alpha(x,y,u,v)\,dx\,dy$ …………………………………….(1.1)

Where the kernel,

$K_s^\alpha(x,y,u,v) = \sqrt{\frac{1-icot\alpha}{2\pi}}\, e^{\frac{i(x^2+y^2+u^2+v^2)cot\alpha}{2}} e^{i((\alpha-\frac{\pi}{2})}$

$\sin(cosec\alpha.ux).\sin(cosec\alpha.vy)$……….. (1.2)

1.2. The test function space E

An infinitely differentiable complex valued function $\emptyset$ on R^n belongs to $E(R^n)$ if for each compact set $I \subset S_{a,b}$, where,

$$S_{a,b} = \{x,y: x,y \in R^n, |x| \le a, |y| \le b, a > 0, b > 0\}, I \in R^n$$

$$\gamma_{E_{p,q}}(\emptyset) = \sup_{x,y}\left|D_{x,y}^{p,q}\emptyset(x,y)\right| < \infty$$

Where, p, q =1, 2, 3….

Thus $E(R^n)$ will denote the space of all $\emptyset \in E(R^n)$ with support contained in $S_{a,b}$

Note: that the space E is complete and therefore a Frechet space. Moreover, we say that f is a fractional Sine transformable, if it is a member of E^*, the dual space of E.

This paper emphasizes to deriving convolution theorem for two dimensional fractional Sine transform and defined distributional two-dimensional fractional Sine transform

2. Distributional two-dimensional fractional Sine transform

The two dimensional distributional fractional Sine transform of $f(x,y) \in E^*(R^n)$ defined by

$F_s^\alpha\{f(x,y)\} = F^\alpha(u,v) = \langle f(x,y), K_\alpha(x,y,u,v)\rangle$ …………………………… (2.1)

$K_s^\alpha(x,y,u,v) = \sqrt{\frac{1-icot\alpha}{2\pi}}\, e^{\frac{i(x^2+y^2+u^2+v^2)cot\alpha}{2}}\, e^{i((\alpha-\frac{\pi}{2})}\sin(cosec\alpha.ux).\sin(cosec\alpha.vy)$ ….. (2.2)

Where , RHS of equation (2.1) has a meaning as the application of $f \in E^*$ to $K_\alpha(x,y,u,v) \in E$

3. Convolution Theorem:

If $h(u,v) = (f.g)(u,v)$ and $FC_\alpha, GC_\alpha, HC_\alpha$ AND $FS_\alpha, GS_\alpha, HS_\alpha$ denotes two dimensional fractional cosine and sine transform f, g, h respectively, then

Proof: From the definition of two dimensional the fractional cosine transform, we have

$FS_\alpha\{f(x,y)\}(u,v) * GS_\alpha\{g(t,s)\}(u,v)$

$$= \left\{ \begin{array}{l} \sqrt{\frac{1-icot\alpha}{2\pi}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\alpha-\frac{\pi}{2})} e^{\frac{i}{2}(x^2+y^2+u^2+v^2)cot\alpha} \\ sin(coseca.ux)sin(coseca.vy)f(x,y)dxdy \\ \sqrt{\frac{1-icot\alpha}{2\pi}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\alpha-\frac{\pi}{2})} e^{\frac{i}{2}(t^2+s^2+u^2+v^2)cot\alpha} \\ sin(coseca.ut)sin(coseca.vs)g(t,s)dtds \end{array} \right\}$$

$FS_\alpha\{f(x,y)\}(u,v) * GS_\alpha\{g(t,s)\}(u,v)$

$$= \left\{ \begin{array}{c} (e^{i(\alpha-\frac{\pi}{2})})^2 \\ (\sqrt{\frac{1-icot\alpha}{2\pi}})^2 e^{i(u^2+v^2)cot\alpha} \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2}(x^2+y^2+t^2+s^2)cot\alpha} \\ sin(coseca.ux)sin(coseca.vy)f(x,y) \\ sin(coseca.ut)sin(coseca.vs)g(t,s)dxdydtds \end{array} \right\}$$

$FS_\alpha\{f(x,y)\}(u,v) * GS_\alpha\{g(t,s)\}(u,v)$

$$= \left\{ \begin{array}{c} (e^{i(\alpha-\frac{\pi}{2})})^2 (\sqrt{\frac{1-icot\alpha}{2\pi}})^2 e^{i(u^2+v^2)cot\alpha} \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2}(x^2+y^2+t^2+s^2)cot\alpha} \\ f(x,y)g(t,s)\frac{-1}{2} \\ \left[\begin{array}{c}\cos(csc\alpha.u(x+t)) \\ -\cos(csc\alpha.u(x-t))\end{array}\right] \\ \frac{-1}{2}\left[\begin{array}{c}\cos(csc\alpha.v(y+s)) \\ -\cos(csc\alpha.v(y-s))\end{array}\right] dxdydtds \end{array} \right\}$$

Let $A = \left(e^{i(\alpha-\frac{\pi}{2})}\right)^2 \left(\sqrt{\frac{1-icot\alpha}{2\pi}}\right)^2$ $B = \frac{e^{i(u^2+v^2)cot\alpha}}{4}$

$FS_\alpha\{f(x,y)\}(u,v) * GS_\alpha\{g(t,s)\}(u,v)$

$$= \left\{ \begin{array}{c} AB\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2}(x^2+y^2+t^2+s^2)cot\alpha} f(x,y)g(t,s) \\ \left[\begin{array}{c} \cos(csc\alpha.u(x+t))\cos(csc\alpha.v(y+s)) - \cos(csc\alpha.u(x+t))\cos(csc\alpha.v(y-s)) \\ -\cos(csc\alpha.u(x-t))\cos(csc\alpha.v(y+s)) + \cos(csc\alpha.u(x-t))\cos(csc\alpha.v(y-s)) \end{array}\right] \\ dxdydtds \end{array} \right\}$$

$FS_\alpha\{f(x,y)\}(u,v) * GS_\alpha\{g(t,s)\}(u,v)$

$= AB\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2}(x^2+y^2+t^2+s^2)cot\alpha} f(x,y)g(t,s)$

$\cos(csc\alpha.u(x+t))\cos(csc\alpha.v(y+s))\,dxdydtds - AB\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2}(x^2+y^2+t^2+s^2)cot\alpha} f(x,y)g(t,s)$

$\cos(csc\alpha.u(x+t))\cos(csc\alpha.v(y-s))\,dxdydtds$

$$- AB\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \begin{array}{c} e^{\frac{i}{2}(x^2+y^2+t^2+s^2)cot\alpha} f(x,y)g(t,s) \\ \cos(csc\alpha.u(x-t))\cos(csc\alpha.v(y+s)) \end{array} dxdydtds$$

$+ AB \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2}(x^2+y^2+t^2+s^2)cot\alpha} f(x,y)g(t,s)$

$\cos(csc\alpha u(x-t))\cos(csc\alpha v(y-s))\, dxdydtds$

$FS_\alpha\{f(x,y)\}(u,v) * GS_\alpha\{g(t,s)\}(u,v) = I_1 - I_2 - I_3 + I_4$ ……………… (3.1)

Let $I_1 = AB \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2}(x^2+y^2+t^2+s^2)cot\alpha} f(x,y)g(t,s)$

$\cos(csc\alpha.u(x+t))\cos(csc\alpha.v(y+s))\, dxdydtds$

Let $x + t = \tau, t = \tau - x,\ y + s = \rho, s = \rho - y\ if\ t = -\infty, \tau = -\infty\ if\ t = \infty, \tau = \infty$

$if\ s = -\infty, \rho = -\infty\ if\ s = \infty, \rho = \infty$

$I_1 = AB \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2}(x^2+y^2)cot\alpha} f(x,y)$

$\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2}((\tau-x)^2+(\rho-y)^2)cot\alpha} g((\tau-x),(\rho-y))d\tau d\rho\}$

$\cos(csc\alpha.u\tau)\cos(csc\alpha.v\rho)\, dxdy$

Let $\overline{g_1(}\ \tau - x, \rho - y) = e^{\frac{i}{2}((\tau-x)^2+(\rho-y)^2)cot\alpha} g((\tau-x),(\rho-y))$

$\overline{f}(x,y) = e^{\frac{i}{2}(x^2+y^2)cot\alpha} f(x,y)$

$I_1 = AB \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{f}(x,y)\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{g_1(}\ \tau - x, \rho - y)d\tau d\rho\}$

$\cos(csc\alpha.u\tau)\cos(csc\alpha.v\rho)\, dxdy$

$I_1 = AB \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{f}(x,y)\overline{g_1(}\ \tau - x, \rho - y)dxdy\}$

$\cos(csc\alpha.v\rho)\cos(csc\alpha.u\tau)\, d\tau d\rho$

$I_1 = (e^{i(\alpha-\frac{\pi}{2})})^2 (\sqrt{\frac{1-icot\alpha}{2\pi}})^2 \frac{e^{i(u^2+v^2)cot\alpha}}{4} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{f}(x,y)\overline{g_1(}\ \tau - x, \rho - y)dxdy\}$

$\cos(csc\alpha.v\rho)\cos(csc\alpha.u\tau)\, d\tau d\rho$

Let $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{f}(x,y)\overline{g_1(}\ \tau - x, \rho - y)dxdy = (\overline{f}.\overline{g_1})$

$I_1 = (e^{i(\alpha-\frac{\pi}{2})})^2 \sqrt{\frac{1-icot\alpha}{2\pi}} \frac{e^{\frac{i}{2}(u^2+v^2)cot\alpha}}{4} e^{\frac{-i}{2}(\tau^2+\rho^2)cot\alpha} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{-i}{2}(\tau^2+\rho^2)cot\alpha} e^{\frac{i}{2}(u^2+v^2+\tau^2+\rho^2)cot\alpha} \sqrt{\frac{1-icot\alpha}{2\pi}} (\overline{f}.\overline{g_1})$

$\cos(csc\alpha.v\rho)\cos(csc\alpha.u\tau)\, d\tau d\rho$

$I_1 = (e^{i(\alpha-\frac{\pi}{2})})^2 \sqrt{\frac{\frac{1-icot\alpha}{2\pi} e^{\frac{i}{2}(u^2+v^2)cot\alpha}}{4}} FC_\alpha\{e^{\frac{-i}{2}(\tau^2+\rho^2)cot\alpha}(\overline{f}.\overline{g_1})\}$

$I_2 = AB \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2}(x^2+y^2+t^2+s^2)cot\alpha} f(x,y)g(t,s)$

$\cos(csc\alpha.u(x+t))\cos(csc\alpha.v(y-s))\, dxdydtds$

$x + t = \tau \;\; y - s = \xi$

Let$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\bar{f}(x,y)\overline{g_2(}\,\tau - x\,,\xi - y)dxdy = (\bar{f}.\overline{g_2})$

For I_2 we do similar calculations in I_1 we get

$$I_2 = (e^{i(\alpha-\frac{\pi}{2})})^2 \sqrt{\frac{1-icot\alpha}{2\pi}}\frac{e^{\frac{i}{2}(u^2+v^2)cot\alpha}}{4} FC_\alpha\{e^{\frac{-i}{2}(\tau^2+\xi^2)cot\alpha}(\bar{f}.\overline{g_2})\}$$

Similarly

$$I_3 = AB\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2}(x^2+y^2+t^2+s^2)cot\alpha} f(x,y)g(t,s)$$

$$\cos(csc\alpha.u(x-t))\cos(csc\alpha.v(y+s))\,dxdydtds$$

Here

$x - t = \eta \;\; y + s = \rho$ Let$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\bar{f}(x,y)\overline{g_3(}\,\eta - x\,,\rho - y)dxdy = (\bar{f}.\overline{g_3})$

$$I_3 = (e^{i(\alpha-\frac{\pi}{2})})^2 \sqrt{\frac{1-icot\alpha}{2\pi}}\frac{e^{\frac{i}{2}(u^2+v^2)cot\alpha}}{4} FC_\alpha\{e^{\frac{-i}{2}(\eta^2+\rho^2)cot\alpha}(\bar{f}.\overline{g_3})\}$$

Similarly

$$I_4 = AB\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2}(x^2+y^2+t^2+s^2)cot\alpha} f(x,y)g(t,s)$$

$$\cos(csc\alpha.u(x-t))\cos(csc\alpha.v(y-s))\,dxdydtds$$

$x - t = \eta \;\; y - s = \xi$ Let$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\bar{f}(x,y)\overline{g_3(}\,\eta - x\,,\xi - y)dxdy = (\bar{f}.\overline{g_4})$

$$I_4 = (e^{i(\alpha-\frac{\pi}{2})})^2 \sqrt{\frac{1-icot\alpha}{2\pi}}\frac{e^{\frac{i}{2}(u^2+v^2)cot\alpha}}{4} FC_\alpha\{e^{\frac{-i}{2}(\eta^2+\xi^2)cot\alpha}(\bar{f}.\overline{g_4})\}$$

Then (3.1) implies

$I = I_1 - I_2 - I_3 + I_4$

$$FS_\alpha\{f(x,y)\}(u,v) * GS_\alpha\{g(t,s)\}(u,v) = (e^{i(\alpha-\frac{\pi}{2})})^2 \sqrt{\frac{1-icot\alpha}{2\pi}}\frac{e^{\frac{i}{2}(u^2+v^2)cot\alpha}}{4} FC_\alpha\{e^{\frac{-i}{2}(\tau^2+\rho^2)cot\alpha}(\bar{f}.\overline{g_1})\}$$
$$-(e^{i(\alpha-\frac{\pi}{2})})^2 \sqrt{\frac{1-icot\alpha}{2\pi}}\frac{e^{\frac{i}{2}(u^2+v^2)cot\alpha}}{4} FC_\alpha\{e^{\frac{-i}{2}(\tau^2+\xi^2)cot\alpha}(\bar{f}.\overline{g_2})\}$$
$$-(e^{i(\alpha-\frac{\pi}{2})})^2 \sqrt{\frac{1-icot\alpha}{2\pi}}\frac{e^{\frac{i}{2}(u^2+v^2)cot\alpha}}{4} FC_\alpha\{e^{\frac{-i}{2}(\eta^2+\rho^2)cot\alpha}(\bar{f}.\overline{g_3})\}$$
$$+(e^{i(\alpha-\frac{\pi}{2})})^2 \sqrt{\frac{1-icot\alpha}{2\pi}}\frac{e^{\frac{i}{2}(u^2+v^2)cot\alpha}}{4} FC_\alpha\{e^{\frac{-i}{2}(\eta^2+\xi^2)cot\alpha}(\bar{f}.\overline{g_4})\}$$

$$FS_\alpha\{f(x,y)\}(u,v) * GS_\alpha\{g(t,s)\}(u,v) = (e^{i(\alpha-\frac{\pi}{2})})^2 \sqrt{\frac{1-icot\alpha}{2\pi}}\frac{e^{\frac{i}{2}(u^2+v^2)cot\alpha}}{4}$$

$$\left[\begin{array}{c} FC_\alpha\{e^{\frac{-i}{2}(\tau^2+\rho^2)cot\alpha}(\bar{f}.\overline{g_1})\} - FC_\alpha\{e^{\frac{-i}{2}(\tau^2+\xi^2)cot\alpha}(\bar{f}.\overline{g_2})\} \\ -FC_\alpha\{e^{\frac{-i}{2}(\eta^2+\rho^2)cot\alpha}(\bar{f}.\overline{g_3})\} + FC_\alpha\{e^{\frac{-i}{2}(\eta^2+\xi^2)cot\alpha}(\bar{f}.\overline{g_4})\} \end{array}\right]$$

Conclusion: Convolution theorem for generalized two dimensional fractional Sine transform is proved.

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