STOCHASTIC ANALYSIS OF SEMI BITENDEM FEEDBACK QUEUE NETWORK CENTRALLY LINKED WITH COMMON CHANNEL

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ABSTRACT

This paper discusses a complex Network of queue in which a common channel is centrally linked in series with each of two systems, one containing two bi-serial sub channels and other containing two parallel sub channels. There is a feedback from a centrally linked channel to each of both sub systems. All the activities in the system concerned are performed under stochastic environment. The arrivals follow Poisson distribution at each channel and service times are distributed exponentially at each channel. The system performance characteristics have been found using statistical formulae, laws of calculus and generating function technique. Numerical illustration has been given to demonstrate the result.

***Keywords**: Steady state behavior, Poisson distribution, Bi-serial channels, Feedback, Generating function, exponential distribution.*

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1. INTRODUCTION:

Jackson (1957, 1963) did a landmark in queue network design and first showed that the solution to steady state balance equations is of product form. Koiengsberg (1958) investigated the buffer storage problem as a system of cyclic queues. A cyclic queue is a sort of series queue in a “circle” where the output of last node feeds back to first node, a special case of a closed queuing network. Finch, P.D. (1959) extended the work of cyclic queues with feedback. Basket etal. (1975) explained multiclass Jackson Networks and obtained product form solutions for three different queue disciplines: (1) processor sharing (each customer gets a share of and is served simultaneously by a single server), (2) ample service, and (3) LCFS with preemptive resume servicing. They allowed the network to be open for some classes of customers and closed for others. Kelley’s work (1976, 79) represented the state of art in the generalization of Jackson networks. Cross and Ince (1981) have applied Kelly’s multiclass results to a closed network and obtained numerical solutions for an application in repairable item inventory control. A remarkable work in the field of queue specially in bi-series was done by Maggu (1970) & Singh T.P.(1986, 2005) etc. Maggu [1970] introduced the concept of bitendem in theory of queues which corresponds to a practical situation arise in production concern. Later on this idea was developed by various researchers with different modifications and augmentations. Singh T.P. etal.[2005] studied the transient behavior of a queuing network with parallel biseries queue linked with a common channel. Singh T.P. etal.[2006] further studied steady state behavior of a queue model comprised of two subsystem with biserial channel linked with a common channel. Later Gupta Deepak, Singh T.P. etal.[2007] studied a network queue model comprised of biserial and parallel channel linked with a common server. Recently Singh T.P., Vinod Kumar (2006, 2007) made an analysis of a queue model comprised of two sub system each having linkage with common channel. Deepak Gupta (2007) studied a complex system of queue network comprised of queue two sub systems each centrally linked with a common server. Singh T.P. & Kusum (2010(a), 2010(b)) extended the work of Gupta Deepak considering feedback queue network under different parameters. Further Singh T.P. & Kusum (2011) studied a different type of heterogeneous feedback queue model & explored a steady state solution. The work has been extended by Arti Tyagi, Singh T.P. etal(2011) with a different angle and feedback queue system has been analyzed in transient form.

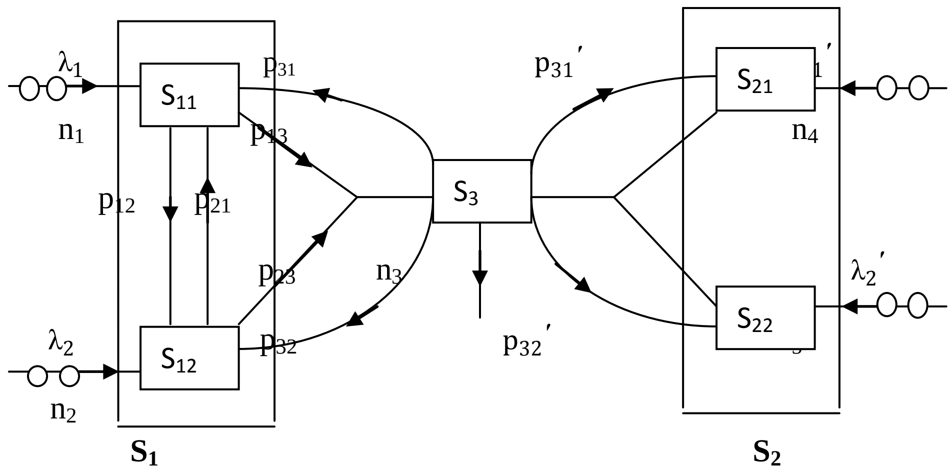
The present queue model differs the study made by above said authors. In this model we assume that the first system consists of biserial channel while second system consists of parallel channel which are linked centrally with third channel in series. There is a feedback service from the common channel to each sub systems. All the concerned activities are being performed under stochastic environment. The formulae and the result derived are significant and have wider applications in Real world situation.

1.1 Application:

It has been observed that in some multistage queuing process recycling or feedback may occur e.g. a telecommunications network may process massages through a randomly selected sequence of nodes with the probability that some messages will require rerouting on occasion through the same stage. The model find its applications in industries, administrative setups, banking system, computer networks, office management, super markets and shopping malls etc. the model is useful in offices and services where different departments are linked for operational purposes with a central authority for final decision making. The defective or incomplete files required more clarification is sent back to initial stages. The model is also useful in a production set up or manufacturing set up where different parts of manufacturing items are being produced in different workshops but for assembly sake they have to pass through the common counter for final assembly and get the desired shape of finished product for selling purpose. The defective items are sent back to different workshops for repair or to change.

2. MODEL DESCRIPTION:

The entire queue model consists of two blocks S_1 and S_2 equipped with a common central service channel S_3. The subsystem S_1 consist of two biserial service channels S_{11} and S_{12}, the block S_2 contains two parallel channels S_{21} and S_{22}. The service time at Sij (i=1,2 and j =1,2) are distributed exponentially. We assume the mean service rate μ_1, μ_2, μ_1' and μ_2' at Sij (i=1,2 and j =1,2) and μ_3 at S_3 respectively. Queues Q_1, Q_2, Q_3, Q_4, Q_5 are said to formed in front of the service channels S_{11}, S_{12}, S_{21}, S_{22} and S_3 respectively, if they are busy. Customers coming at the rate λ_1 after completion of phase service at S_{11} will join S_{12} or S_3 (that is they may either go to the network of servers $S_{11} \rightarrow S_{12} \rightarrow S_3$ or $S_{11} \rightarrow S_3$) with the probabilities p_{12} or p_{13} such that $p_{12} + p_{13} = 1$ □ and those coming at the rate λ_2 after completion of phase service at S_{12} will join S_{11} or S_3 (that is they may either go to the network of servers $S_{21} \rightarrow S_{11} \rightarrow S_3$ or $S_{21} \rightarrow S_3$) with the probabilities p_{21} or p_{23} such that $p_{21}+p_{23} = 1$. The customers coming at the rate λ_1' go to the network of servers $S_{21} \rightarrow S_3$ and those coming at the rate λ_2' go to the network of servers $S_{22} \rightarrow S_3$. There is feedback from server S_3 to S_{11}, S_{12}, S_{21}, S_{22} with the probabilities p_{31}, p_{32}, p_{31}', p_{32}' respectively, such that $p_{31} + p_{32} + p_{31}' + p_{32}' = 1$



3. MATHEMATICAL ANALYSIS OF QUEUE MODEL:-

Define $P_{n_1, n_2, n_3, n_4, n_5}$ be the joint probability that there are n_1 units waiting in queue Q_1 in front of S_{11}, n_2 units waiting in queue Q_2 in front of S_{12}, n_3 units waiting in queue Q_3 in front of S_3, n_4 units waiting in queue Q_4 in front of S_{21}, n_5 units waiting in queue Q_5 in front of S_{22} In each case the waiting includes a unit in service, if any. Also, $n_1, n_2, n_3, n_4, n_5 > 0$.

The standard argument leads to the following differential difference equations in steady state –

$$(\lambda_1 + \lambda_2 + \mu_1 + \mu_2 + \mu_3 + \mu_1' + \mu_2' + \lambda_1' + \lambda_2')P_{n_1,n_2,n_3,n_4,n_5} =$$
$$\lambda_1 P_{n_1-1,n_2,n_3,n_4,n_5} + \lambda_2 P_{n_1,n_2-1,n_3,n_4,n_5} + \lambda_1' P_{n_1,n_2,n_3,n_4-1,n_5} +$$
$$\lambda_2' P_{n_1,n_2,n_3,n_4,n_5-1} + \mu_1 p_{12} P_{n_1+1,n_2-1,n_3,n_4,n_5} + \mu_1 p_{13} P_{n_1+1,n_2,n_3-1,n_4,n_5} + \mu_2 p_{21} P_{n_1-1,n_2+1,n_3,n_4,n_5} +$$
$$\mu_2 p_{23} P_{n_1,n_2+1,n_3-1,n_4,n_5} + \mu_3 p_{31} P_{n_{1-1},n_2,n_3+1,n_4,n_5} + \mu_3 p_{32} P_{n_1,n_2-1,n_3+1,n_4,n_5} + \mu_1' P_{n_1,n_2,n_3-1,n_4+1,n_5}$$
$$+ \mu_2' P_{n_1,n_2,n_3-1,n_4,n_5+1} + \mu_3 p_{31}' P_{n_1,n_2,n_3+1,n_4-1,n_5} + \mu_3 p_{32}' P_{n_1,n_2,n_3+1,n_4,n_5-1} + \mu_3 P_{n_1,n_2,n_3+1,n_4,n_5}$$

$$n_1, n_2, n_3, n_4, n_5 \geq 0 \qquad (1)$$

$$(\lambda_1 + \lambda_2 + \mu_1 + \mu_2 + \mu_3 + \mu_1' + \mu_2' + \lambda_1' + \lambda_2')P_{0,n_2,n_3,n_4,n_5} = \lambda_2 P_{0,n_2-1,n_3,n_4,n_5} +$$
$$\lambda_1' P_{0,n_2,n_3,n_4-1,n_5} + \lambda_2' P_{0,n_2,n_3,n_4,n_5-1} + \mu_1 p_{12} P_{1,n_2-1,n_3,n_4,n_5} + \mu_1 p_{13} P_{1,n_2,n_3-1,n_4,n_5} +$$
$$\mu_2 p_{23} P_{0,n_2+1,n_3-1,n_4,n_5} + \mu_3 p_{32} P_{0,n_2-1,n_3+1,n_4,n_5} + \mu_1' P_{0,n_2,n_3-1,n_4+1,n_5} + \mu_2' P_{0,n_2,n_3-1,n_4,n_5+1} +$$
$$\mu_3 p_{31}' P_{0,n_2,n_3+1,n_4-1,n_5} + \mu_3 p_{32}' P_{0,n_2,n_3+1,n_4,n_5-1} + \mu_3 P_{0,n_2,n_3+1,n_4,n_5}$$

$$n_1 = 0, n_2, n_3, n_4, n_5 \geq 0 \qquad (2)$$

$$(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_1{}'+\mu_2{}'+\lambda_1{}'+\lambda_2{}')P_{n_1,0,n_3,n_4,n_5} = \lambda_1 P_{n_1-1,0,n_3,n_4,n_5} + \lambda_1{}' P_{n_1,0,n_3,n_4-1,n_5} + \lambda_2{}' P_{n_1,0,n_3,n_4,n_5-1} + \mu_1 p_{13} P_{n_1+1,0,n_3-1,n_4,n_5} + \mu_2 p_{21} P_{n_1-1,1,n_3,n_4,n_5} + \mu_2 p_{23} P_{n_1,1,n_3-1,n_4,n_5} + \mu_3 p_{31} P_{n_1-1,0,n_3+1,n_4,n_5} + \mu_1{}' P_{n_1,0,n_3-1,n_4+1,n_5} + \mu_2{}' P_{n_1,0,n_3-1,n_4,n_5+1} + \mu_3 p_{31}{}' P_{n_1,0,n_3+1,n_4-1,n_5} + \mu_3 p_{32}{}' P_{n_1,0,n_3+1,n_4,n_5-1} + \mu_3 P_{n_1,0,n_3+1,n_4,n_5}$$

$$n_2 = 0,\ n_1 n_3, n_4, n_5 \geq 0 \qquad (3)$$

$$(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_1{}'+\mu_2{}'+\lambda_1{}'+\lambda_2{}')P_{n_1,n_2,0,n_4,n_5} = \lambda_1 P_{n_1-1,n_2,0,n_4,n_5} + \lambda_2 P_{n_1,n_2-1,0,n_4,n_5} + \lambda_1{}' P_{n_1,n_2,0,n_4-1,n_5} + \lambda_2{}' P_{n_1,n_2,0,n_4,n_5-1} + \mu_1 p_{12} P_{n_1+1,n_2-1,0,n_4,n_5} + \mu_2 p_{21} P_{n_1-1,n_2+1,0,n_4,n_5} + \mu_3 p_{31} P_{n_{1-1},n_2,1,n_4,n_5} + \mu_3 p_{32} P_{n_1,n_2-1,1,n_4,n_5} + \mu_3 p_{31}{}' P_{n_1,n_2,1,n_4-1,n_5} + \mu_3 p_{32}{}' P_{n_1,n_2,1,n_4,n_5-1} + \mu_3 P_{n_1,n_2,1,n_4,n_5}$$

$$n_3 = 0,\ n_2, n_1, n_4, n_5 \geq 0 \qquad (4)$$

$$(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_1{}'+\mu_2{}'+\lambda_1{}'+\lambda_2{}')P_{n_1,n_2,n_3,0,n_5} = \lambda_1 P_{n_1-1,n_2,n_3,0,n_5} + \lambda_2 P_{n_1,n_2-1,n_3,0,n_5} + \lambda_2{}' P_{n_1,n_2,n_3,0,n_5-1} + \mu_1 p_{12} P_{n_1+1,n_2-1,n_3,0,n_5} + \mu_1 p_{13} P_{n_1+1,n_2,n_3-1,0,n_5} + \mu_2 p_{21} P_{n_1-1,n_2+1,n_3,0,n_5} + \mu_2 p_{23} P_{n_1,n_2+1,n_3-1,0,n_5} + \mu_3 p_{31} P_{n_{1-1},n_2,n_3+1,0,n_5} + \mu_3 p_{32} P_{n_1,n_2-1,n_3+1,0,n_5} + \mu_1{}' P_{n_1,n_2,n_3-1,1,n_5} + \mu_2{}' P_{n_1,n_2,n_3-1,0,n_5+1} + \mu_3 p_{32}{}' P_{n_1,n_2,n_3+1,0,n_5-1} + \mu_3 P_{n_1,n_2,n_3+1,0,n_5}$$

$$n_4 = 0,\ n_2, n_1, n_3, n_5 \geq 0 \qquad (5)$$

$$(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_1{}'+\mu_2{}'+\lambda_1{}'+\lambda_2{}')P_{n_1,n_2,n_3,n_4,0} = \lambda_1 P_{n_1-1,n_2,n_3,n_4,0} + \lambda_2 P_{n_1,n_2-1,n_3,n_4,0} + \lambda_1{}' P_{n_1,n_2,n_3,n_4-1,0} + \mu_1 p_{12} P_{n_1+1,n_2-1,n_3,n_4,0} + \mu_1 p_{13} P_{n_1+1,n_2,n_3-1,n_4,0} + \mu_2 p_{21} P_{n_1-1,n_2+1,n_3,n_4,0} + \mu_2 p_{23} P_{n_1,n_2+1,n_3-1,n_4,0} + \mu_3 p_{31} P_{n_{1-1},n_2,n_3+1,n_4,0} + \mu_3 p_{32} P_{n_1,n_2-1,n_3+1,n_4,0} + \mu_1{}' P_{n_1,n_2,n_3-1,n_4+1,0} + \mu_2{}' P_{n_1,n_2,n_3-1,n_4,1} + \mu_3 p_{31}{}' P_{n_1,n_2,n_3+1,n_4-1,0} + \mu_3 P_{n_1,n_2,n_3+1,n_4,0}$$

$$n_5 = 0,\ n_2, n_1, n_3, n_4 \geq 0 \qquad (6)$$

$$(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_1{}'+\mu_2{}'+\lambda_1{}'+\lambda_2{}')P_{0,0,n_3,n_4,n_5} = \lambda_1{}' P_{0,0,n_3,n_4-1,n_5} + \lambda_2{}' P_{0,0,n_3,n_4,n_5-1} + \mu_1 p_{13} P_{1,0,n_3-1,n_4,n_5} + \mu_2 p_{23} P_{0,1,n_3-1,n_4,n_5} + \mu_1{}' P_{0,0,n_3-1,n_4+1,n_5} + \mu_2{}' P_{0,0,n_3-1,n_4,n_5+1} + \mu_3 p_{31}{}' P_{0,0,n_3+1,n_4-1,n_5} + \mu_3 p_{32}{}' P_{0,0,n_3+1,n_4,n_5-1} + \mu_3 P_{0,0,n_3+1,n_4,n_5}$$

$$n_1, n_2 = 0,\ n_3, n_4, n_5 \geq 0 \qquad (7)$$

$$(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_1{}'+\mu_2{}'+\lambda_1{}'+\lambda_2{}')P_{0,n_2,0,n_4,n_5} = \lambda_2 P_{0,n_2-1,0,n_4,n_5} + \lambda_1{}' P_{0,n_2,0,n_4-1,n_5} + \lambda_2{}' P_{0,n_2,0,n_4,n_5-1} + \mu_1 p_{12} P_{1,n_2-1,0,n_4,n_5} + \mu_3 p_{32} P_{0,n_2-1,1,n_4,n_5} + \mu_3 p_{31}{}' P_{0,n_2,1,n_4-1,n_5} + \mu_3 p_{32}{}' P_{0,n_2,1,n_4,n_5-1} + \mu_3 P_{0,n_2,1,n_4,n_5}$$

$$n_1, n_3 = 0,\ n_2,, n_4, n_5 \geq 0 \qquad (8)$$

$$(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_1{}'+\mu_2{}'+\lambda_1{}'+\lambda_2{}')P_{0,n_2,n_3,0,n_5} = \lambda_2 P_{0,n_2-1,n_3,0,n_5} + \lambda_2{}' P_{0,n_2,n_3,0,n_5-1} + \mu_1 p_{12} P_{1,n_2-1,n_3,0,n_5} + \mu_1 p_{13} P_{1,n_2,n_3-1,0,n_5} + \mu_2 p_{23} P_{0,n_2+1,n_3-1,0,n_5} + \mu_3 p_{32} P_{0,n_2-1,n_3+1,0,n_5} + \mu_1{}' P_{0,n_2,n_3-1,0,1,n_5} + \mu_2{}' P_{0,n_2,n_3-1,0,n_5+1} + \mu_3 p_{32}{}' P_{0,n_2,n_3+1,0,n_5-1} + \mu_3 P_{0,n_2,n_3+1,0,n_5}$$

$$n_1, n_4, = 0\ , n_2, n_3, n_5 \geq 0 \qquad (9)$$

$$(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_1{}'+\mu_2{}'+\lambda_1{}'+\lambda_2{}')P_{0,n_2,n_3,n_4,0} = \lambda_2 P_{0,n_2-1,n_3,n_4,0} + \lambda_1{}' P_{0,n_2,n_3,n_4-1,0} + \mu_1 p_{12} P_{1,n_2-1,n_3,n_4,0} + \mu_1 p_{13} P_{1,n_2,n_3-1,n_4,0} + \mu_2 p_{23} P_{0,n_2+1,n_3-1,n_4,0} + \mu_3 p_{32} P_{0,n_2-1,n_3+1,n_4,0} + \mu_1{}' P_{0,n_2,n_3-1,n_4+1,0} + \mu_2{}' P_{0,n_2,n_3-1,n_4,1} + \mu_3 p_{31}{}' P_{0,n_2,n_3+1,n_4-1,0} + \mu_3 P_{0,n_2,n_3+1,n_4,0}$$

$$n_1, n_5, = 0\ , n_2, n_3, n_4 \geq 0 \qquad (10)$$

$$(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_1{}'+\mu_2{}'+\lambda_1{}'+\lambda_2{}')P_{n_1,0,0,n_4,n_5} = \lambda_1 P_{n_1-1,0,0,n_4,n_5} + \lambda_1{}' P_{n_1,0,0,n_4-1,n_5} + \lambda_2{}' P_{n_1,0,0,n_4,n_5-1} + \mu_2 p_{21} P_{n_1-1,1,0,n_4,n_5} + \mu_3 p_{31} P_{n_1-1,0,1,n_4,n_5} + \mu_3 p_{31}{}' P_{n_1,0,1,n_4-1,n_5} + \mu_3 p_{32}{}' P_{n_1,0,1,n_4,n_5-1} + \mu_3 P_{n_1,0,1,n_4,n_5}$$

$$n_2, n_3 = 0\ , n_1, n_4, n_5 \geq 0 \qquad (11)$$

$$(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_1{}'+\mu_2{}'+\lambda_1{}'+\lambda_2{}')P_{n_1,0,n_3,0,n_5} = \lambda_1 P_{n_1-1,0,n_3,0,n_5} + \lambda_2{}' P_{n_1,0,n_3,0,n_5-1} + \mu_1 p_{13} P_{n_1+1,0,n_3-1,0,n_5} + \mu_2 p_{21} P_{n_1-1,1,n_3,0,n_5} + \mu_2 p_{23} P_{n_1,1,n_3-1,0,n_5} + \mu_3 p_{31} P_{n_1-1,0,n_3+1,0,n_5} + \mu_1{}' P_{n_1,0,n_3-1,1,n_5} + \mu_2{}' P_{n_1,0,n_3-1,0,n_5+1} + \mu_3 p_{32}{}' P_{n_1,0,n_3+1,0,n_5-1} + \mu_3 P_{n_1,0,1,n_4,n_5}$$

$$n_2, n_4 = 0\ , n_1 n_3, n_5 \geq 0 \qquad (12)$$

$$(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_1{}'+\mu_2{}'+\lambda_1{}'+\lambda_2{}')P_{n_1,0,n_3,n_4,0} = \lambda_1 P_{n_1-1,0,n_3,n_4,0} + \lambda_1{}' P_{n_1,0,n_3,n_4-1,0} + \mu_1 p_{13} P_{n_1+1,0,n_3-1,n_4,0} + \mu_2 p_{21} P_{n_1-1,1,n_3,n_4,0} + \mu_2 p_{23} P_{n_1,1,n_3-1,n_4,0} + \mu_3 p_{31} P_{n_1-1,0,n_3+1,n_4,0} + \mu_1{}' P_{n_1,0,n_3-1,n_4+1,0} + \mu_2{}' P_{n_1,0,n_3-1,n_4,1} + \mu_3 p_{31}{}' P_{n_1,0,n_3+1,n_4-1,0} + \mu_3 P_{n_1,0,n_3+1,n_4,0}$$

$$n_2, n_5 = 0\ , n_1 n_3, n_4 \geq 0 \qquad (13)$$

$$(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_1{}'+\mu_2{}'+\lambda_1{}'+\lambda_2{}')P_{n_1,n_2,0,0,n_5} = \lambda_1 P_{n_1-1,n_2,0,0,n_5} + \lambda_2 P_{n_1,n_2-1,0,0,n_5} + \lambda_2{}' P_{n_1,n_2,0,0,n_5-1} + \mu_1 p_{12} P_{n_1+1,n_2-1,0,0,n_5} + \mu_2 p_{21} P_{n_1-1,n_2+1,0,0,n_5} + \mu_3 p_{31} P_{n_{1-1},n_2,1,0,n_5} + \mu_3 p_{32} P_{n_1,n_2-1,1,0,n_5} + \mu_3 p_{32}{}' P_{n_1,n_2,1,0,n_5-1} + \mu_3 P_{n_1,n_2,1,0,n_5}$$

$$n_3, n_4 = 0, n_2, n_1, n_5 \geq 0 \qquad (14)$$

$$(\lambda_1 + \lambda_2 + \mu_1 + \mu_2 + \mu_3 + \mu_1' + \mu_2' + \lambda_1' + \lambda_2')P_{n_1,n_2,0,n_4,0} = \lambda_1 P_{n_1-1,n_2,0,n_4,0} + \lambda_2 P_{n_1,n_2-1,0,n_4,0} + \lambda_1' P_{n_1,n_2,0,n_4-1,0} + \mu_1 p_{12} P_{n_1+1,n_2-1,0,n_4,0} + \mu_2 p_{21} P_{n_1-1,n_2+1,0,n_4,0} + \mu_3 p_{31} P_{n_{1-1},n_2,1,n_4,0} + \mu_3 p_{32} P_{n_1,n_2-1,1,n_4,0} + \mu_3 p_{31}' P_{n_1,n_2,1,n_4-1,0} + \mu_3 P_{n_1,n_2,1,n_4,0}$$

$$n_3, n_5 = 0, n_1, n_2, n_4 \geq 0 \qquad (15)$$

$$(\lambda_1 + \lambda_2 + \mu_1 + \mu_2 + \mu_3 + \mu_1' + \mu_2' + \lambda_1' + \lambda_2')P_{n_1,n_2,n_3,0,0} = \lambda_1 P_{n_1-1,n_2,n_3,0,0} + \lambda_2 P_{n_1,n_2-1,n_3,0,0} + \mu_1 p_{12} P_{n_1+1,n_2-1,n_3,0,0} + \mu_1 p_{13} P_{n_1+1,n_2,n_3-1,0,0} + \mu_2 p_{21} P_{n_1-1,n_2+1,n_3,0,0} + \mu_2 p_{23} P_{n_1,n_2+1,n_3-1,0,0} + \mu_3 p_{31} P_{n_{1-1},n_2,n_3+1,00} + \mu_3 p_{32} P_{n_1,n_2-1,n_3+1,0,0} + \mu_1' P_{n_1,n_2,n_3-1,1,0} + \mu_2' P_{n_1,n_2,n_3-1,0,1} + \mu_3 P_{n_1,n_2,n_3+1,0,0}$$

$$n_4, n_5 = 0, n_1, n_2, n_3 \geq 0 \qquad (16)$$

$$(\lambda_1 + \lambda_2 + \mu_1 + \mu_2 + \mu_3 + \mu_1' + \mu_2' + \lambda_1' + \lambda_2')P_{0,0,0,n_4,n_5} = \lambda_1' P_{0,0,0,n_4-1,n_5} + \lambda_2' P_{0,0,0,n_4,n_5-1} + \mu_3 p_{31}' P_{0,0,1,n_4-1,n_5} + \mu_3 p_{32}' P_{0,0,1,n_4,n_5-1} + \mu_3 P_{0,0,+1,n_4,n_5}$$

$$n_1, n_2, n_3 = 0, n_4, n_5 \geq 0 \qquad (17)$$

$$(\lambda_1 + \lambda_2 + \mu_1 + \mu_2 + \mu_3 + \mu_1' + \mu_2' + \lambda_1' + \lambda_2')P_{0,0,n_3,0,n_5} = \lambda_2' P_{0,0,n_3,0,n_5-1} + \mu_1 p_{13} P_{1,0,n_3-1,0,n_5} + \mu_2 p_{23} P_{0,1,n_3-1,0,n_5} + \mu_1' P_{0,0,n_3-1,1,n_5} + \mu_2' P_{0,0,n_3-1,0,n_5+1} + \mu_3 p_{32}' P_{0,0,n_3+1,0,n_5-1} + \mu_3 P_{0,0,n_3+1,0,n_5}$$

$$n_1, n_2, n_4 = 0, n_3, n_5 \geq 0 \qquad (18)$$

$$(\lambda_1 + \lambda_2 + \mu_1 + \mu_2 + \mu_3 + \mu_1' + \mu_2' + \lambda_1' + \lambda_2')P_{0,0,n_3,n_4,0} = \lambda_1' P_{0,0,n_3,n_4-1,0} + \mu_1 p_{13} P_{1,0,n_3-1,n_4,0} + \mu_2 p_{23} P_{0,1,n_3-1,n_4,0} + \mu_1' P_{0,0,n_3-1,n_4+1,0} + \mu_2' P_{0,0,n_3-1,n_4,1} + \mu_3 p_{31}' P_{0,0,n_3+1,n_4-1,0} + \mu_3 P_{0,0,n_3+1,n_4,0}$$

$$n_1, n_2, n_5 = 0, n_3, n_4 \geq 0 \qquad (19)$$

$$(\lambda_1 + \lambda_2 + \mu_1 + \mu_2 + \mu_3 + \mu_1' + \mu_2' + \lambda_1' + \lambda_2')P_{0,n_2,0,0,n_5} = \lambda_2 P_{0,n_2-1,0,0,n_5} + \lambda_2' P_{0,n_2,0,0,n_5-1} + \mu_1 p_{12} P_{1,n_2-1,0,0,n_5} + \mu_3 p_{32} P_{0,n_2-1,1,0,n_5} + \mu_3 p_{32}' P_{0,n_2,1,0,n_5-1} + \mu_3 P_{0,n_2,1,0,n_5}$$

$$n_1, n_3, n_4 = 0\ , n_2, , n_4, n_5 \geq 0 \qquad (20)$$

$$(\lambda_1 + \lambda_2 + \mu_1 + \mu_2 + \mu_3 + \mu_1{}' + \mu_2{}' + \lambda_1{}' + \lambda_2{}')P_{0,n_2,0,n_4,0} = \lambda_2 P_{0,n_2-1,0,n_4,0} + \lambda_1{}' P_{0,n_2,0,n_4-1,0} + \mu_1 p_{12} P_{1,n_2-1,0,n_4,0} + \mu_3 p_{32} P_{0,n_2-1,1,n_4,0} + \mu_3 p_{31}{}' P_{0,n_2,1,n_4-1,0} + \mu_3 P_{0,n_2,1,n_4,0}$$

$$n_1, n_3, n_5 = 0\ , n_2, , n_4 \geq 0 \qquad (21)$$

$$(\lambda_1 + \lambda_2 + \mu_1 + \mu_2 + \mu_3 + \mu_1{}' + \mu_2{}' + \lambda_1{}' + \lambda_2{}')P_{0,n_2,n_3,0,0} = \lambda_2 P_{0,n_2-1,n_3,0,0} + \mu_1 p_{13} P_{1,n_2,n_3-1,0,0} + \mu_1 p_{12} P_{1,n_2-1,n_3,0,0} + \mu_2 p_{23} P_{0,n_2+1,n_3-1,0,0} + \mu_3 p_{32} P_{0,n_2-1,n_3+1,0,0} + \mu_1{}' P_{0,n_2,n_3-1,0,1,0} + \mu_2{}' P_{0,n_2,n_3-1,0,1} + \mu_3 P_{0,n_2,n_3+1,0,0}$$

$$n_1, n_4, n_5 = 0\ , n_2, n_3, \geq 0 \qquad (22)$$

$$(\lambda_1 + \lambda_2 + \mu_1 + \mu_2 + \mu_3 + \mu_1{}' + \mu_2{}' + \lambda_1{}' + \lambda_2{}')P_{n_1,0,0,0,n_5} = \lambda_1 P_{n_1-1,0,0,0,n_5} + \lambda_2{}' P_{n_1,0,0,0,n_5-1} + \mu_2 p_{21} P_{n_1-1,1,0,0,n_5} + \mu_3 p_{31} P_{n_1-1,0,1,0,n_5} + \mu_3 p_{32}{}' P_{n_1,0,1,0,n_5-1} + \mu_3 P_{n_1,0,1,0,n_5}$$

$$n_2, n_3, n_4 = 0\ , n_1, n_5 \geq 0 \qquad (23)$$

$$(\lambda_1 + \lambda_2 + \mu_1 + \mu_2 + \mu_3 + \mu_1{}' + \mu_2{}' + \lambda_1{}' + \lambda_2{}')P_{n_1,0,0,n_4,0} = \lambda_1 P_{n_1-1,0,0,n_4,0} + \lambda_1{}' P_{n_1,0,0,n_4-1,0} + \mu_2 p_{21} P_{n_1-1,1,0,n_4,0} + \mu_3 p_{31} P_{n_1-1,0,1,n_4,0} + \mu_3 p_{31}{}' P_{n_1,0,1,n_4-1,0} + \mu_3 P_{n_1,0,1,n_4,0} \qquad n_2, n_3, n_5 = 0\ , n_1, n_4 \geq 0 \qquad (24)$$

$$(\lambda_1 + \lambda_2 + \mu_1 + \mu_2 + \mu_3 + \mu_1{}' + \mu_2{}' + \lambda_1{}' + \lambda_2{}')P_{n_1,0,n_3,0,0} = \lambda_1 P_{n_1-1,0,n_3,0,0} + \mu_1 p_{13} P_{n_1+1,0,n_3-1,0,0} + \mu_2 p_{21} P_{n_1-1,1,n_3,0,0} + \mu_2 p_{23} P_{n_1,1,n_3-1,0,0} + \mu_3 p_{31} P_{n_1-1,0,n_3+1,0,0} + \mu_1{}' P_{n_1,0,n_3-1,1,0} + \mu_2{}' P_{n_1,0,n_3-1,0,1} + \mu_3 P_{n_1,0,1,n_4,0}$$

$$n_2, n_4 n_5 = 0\ , n_1 n_3, \geq 0 \qquad (25)$$

$$(\lambda_1 + \lambda_2 + \mu_1 + \mu_2 + \mu_3 + \mu_1{}' + \mu_2{}' + \lambda_1{}' + \lambda_2{}')P_{n_1,n_2,0,0,0} = \lambda_1 P_{n_1-1,n_2,0,0,n_5} + \lambda_2 P_{n_1,n_2-1,0,0,0} + \mu_1 p_{12} P_{n_1+1,n_2-1,0,0,0} + \mu_2 p_{21} P_{n_1-1,n_2+1,0,0,0} + \mu_3 p_{31} P_{n_{1-1},n_2,1,0,0} + \mu_3 p_{32} P_{n_1,n_2-1,1,0,0} + \mu_3 P_{n_1,n_2,1,0,0} \qquad n_3, n_4, n_5 = 0\ , n_2, n_1 \geq 0 \qquad (26)$$

$$(\lambda_1 + \lambda_2 + \mu_1 + \mu_2 + \mu_3 + \mu_1{}' + \mu_2{}' + \lambda_1{}' + \lambda_2{}')P_{0,0,0,0,n_5} = \lambda_2{}' P_{0,0,0,0,n_5-1} + \mu_3 p_{32}{}' P_{0,0,1,0,n_5-1} + \mu_3 P_{0,0,1,0,n_5} \qquad n_1, n_2, n_3, n_4 = 0\ , n_5 \geq 0 \qquad (27)$$

$$(\lambda_1 + \lambda_2 + \mu_1 + \mu_2 + \mu_3 + \mu_1{}' + \mu_2{}' + \lambda_1{}' + \lambda_2{}')P_{0,0,0,n_4,0} = \lambda_1{}' P_{0,0,0,n_4-1,0} + \mu_3 p_{31}{}' P_{0,0,1,n_4-1,0} + \mu_3 P_{0,0,+1,n_4,0}$$

$$n_1, n_2, n_3, n_5 = 0\ , n_4 \geq 0 \qquad (28)$$

$(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_1'+\mu_2'+\lambda_1'+\lambda_2')P_{n_1,0,0,0,0} =$
$\lambda_1 P_{n_1-1,0,0,0,0} + \mu_2 p_{21} P_{n_1-1,1,0,0,0} + \mu_3 p_{31} P_{n_1-1,0,1,0,0} + \mu_3 P_{n_1,0,1,0,0}$

$$n_2, n_3, n_4, n_5 = 0, n_1 \geq 0 \qquad (29)$$

$(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_1'+\mu_2'+\lambda_1'+\lambda_2')P_{0,0,n_3,0,0} =$
$\mu_1 p_{13} P_{1,0,n_3-1,0,0} + \mu_2 p_{23} P_{0,1,n_3-1,0,0} + \mu_1' P_{0,0,n_3-1,1,0} + \mu_2' P_{0,0,n_3-1,0,1} + \mu_3 P_{0,0,n_3+1,0,}$

$$n_1, n_2, n_4, n_5 = 0, n_3, n_5 \geq 0 \qquad (30)$$

$(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_1'+\mu_2'+\lambda_1'+\lambda_2')P_{0,n_2,0,0,0} =$
$\lambda_2 P_{0,n_2-1,0,0,0} + \mu_1 p_{12} P_{1,n_2-1,0,0,0} + \mu_3 p_{32} P_{0,n_2-1,1,0,0} + \mu_3 P_{0,n_2,1,0,0}$

$$n_1, n_3, n_5 = 0, n_2,, n_4 \geq 0 \qquad (31)$$

$(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_1'+\mu_2'+\lambda_1'+\lambda_2')P_{0,0,0,0,n_5} = \mu_3 P_{0,0,1,0,0}$

$$n_1, n_2, n_3, n_4, n_5 = 0 \qquad (32)$$

4. SOLUTION METHODOLOGY:

To solve the system of equations we apply generating function technique.

For this define g.f. as :

$F(X,Y,Z,R,S) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\sum_{n_4=0}^{\infty}\sum_{n_5=0}^{\infty} P_{n_1,n_2,n_3,n_4,n_5} X^{n_1} Y^{n_2} Z^{n_3} R^{n_4} S^{n_5}$

$F_{n_2,n_3,n_4,n_5}(X) = \sum_{n_1=0}^{\infty} P_{n_1,n_2,n_3,n_4,n_5} X^{n_1}$

$F_{n_3,n_4,n_5}(X,Y) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} F_{n_2,n_3,n_4,n_5}(X)\ Y^{n_2}$

$F_{n_4,n_5}(X,Y,Z) = \sum_{n_3=0}^{\infty} F_{n_3}(X,Y)\ Z^{n_3}$

$F_{n_5}(X,Y,Z,R) = \sum_{n_4=0}^{\infty} F_{n_4,n_5}(X,Y,Z)\ R^{n_4}$

$F(X,Y,Z,R,S) = \sum_{n_5=0}^{\infty} F_{n_5}(X,Y,Z,R)\ S^{n_5} \qquad (33)$

On solving equations from (1) to (32) with the help of (33) we get,

$$\Big[\lambda_1(1-X)+\lambda_2(1-Y)+\lambda_1{}'(1-R)+\lambda_2{}'(1-S)+\mu_1(1-\frac{p_{12}Y}{X}-\frac{p_{13}Z}{X})+\mu_2(1-\frac{p_{21}X}{Y}-\frac{p_{23}Z}{Y})+\mu_3(1-\frac{1}{Z}-\frac{p_{31}X}{Z}-\frac{p_{32}Y}{Z}-\frac{p'_{31}R}{Z}-\frac{p'_{32}S}{Z})+\mu_1{}'(1-\frac{Z}{R})+\mu_2{}'(1-\frac{Z}{S})\Big]F(X,Y,Z,R,S)=$$

$$\Big[\mu_1\left(1-\frac{p_{12}Y}{X}-\frac{p_{13}Z}{X}\right)F(Y,Z,R,S)+\mu_2\left(1-\frac{p_{21}X}{Y}-\frac{p_{23}Z}{Y}\right)F(X,Z,R,S)+\mu_1{}'\left(1-\frac{Z}{R}\right)F(X,Y,Z,S)+\mu_2{}'(1-\frac{Z}{S})F(X,Y,Z,R)+\mu_3(1-\frac{1}{Z}-\frac{p_{31}X}{Z}-\frac{p_{32}Y}{Z}-\frac{p'_{31}R}{Z}-\frac{p'_{32}S}{Z})F(X,Y,R,S)\Big]$$

$$F(X,Y,Z,R,S)=\frac{\begin{bmatrix}\mu_1\left(1-\frac{p_{12}Y}{X}-\frac{p_{13}Z}{X}\right)F(Y,Z,R,S)+\mu_2\left(1-\frac{p_{21}X}{Y}-\frac{p_{23}Z}{Y}\right)F(X,Z,R,S)+\mu_1{}'\left(1-\frac{Z}{R}\right)F(X,Y,Z,S)\\+\mu_2{}'\left(1-\frac{Z}{S}\right)F(X,Y,Z,R)+\mu_3\left(1-\frac{1}{Z}-\frac{p_{31}X}{Z}-\frac{p_{32}Y}{Z}-\frac{p'_{31}R}{Z}-\frac{p'_{32}S}{Z}\right)F(X,Y,R,S)\end{bmatrix}}{\begin{bmatrix}\lambda_1(1-X)+\lambda_2(1-Y)+\lambda_1{}'(1-R)+\lambda_2{}'(1-S)+\mu_1\left(1-\frac{p_{12}Y}{X}-\frac{p_{13}Z}{X}\right)+\mu_2\left(1-\frac{p_{21}X}{Y}-\frac{p_{23}Z}{Y}\right)\\+\mu_3\left(1-\frac{1}{Z}-\frac{p_{31}X}{Z}-\frac{p_{32}Y}{Z}-\frac{p'_{31}R}{Z}-\frac{p'_{32}S}{Z}\right)+\mu_1{}'\left(1-\frac{Z}{R}\right)+\mu_2{}'\left(1-\frac{Z}{S}\right)\end{bmatrix}}$$

(34)

For Convenience, let us denote

$F(Y,Z,R,S)=F_1$, $F(X,Z,R,S)=F_2$, $F(X,Y,Z,S)=F_3$, $F(X,Y,Z,R)=F_4$, $F(X,Y,R,S)=F_5$

$$F(X,Y,Z,R,S)=\frac{\begin{bmatrix}\mu_1\left(1-\frac{p_{12}Y}{X}-\frac{p_{13}Z}{X}\right)F_1+\mu_2\left(1-\frac{p_{21}X}{Y}-\frac{p_{23}Z}{Y}\right)F_2+\mu_1{}'\left(1-\frac{Z}{R}\right)F_4\\+\mu_2{}'\left(1-\frac{Z}{S}\right)F_5+\mu_3\left(1-\frac{1}{Z}-\frac{p_{31}X}{Z}-\frac{p_{32}Y}{Z}-\frac{p'_{31}R}{Z}-\frac{p'_{32}S}{Z}\right)F_3\end{bmatrix}}{\begin{bmatrix}\lambda_1(1-X)+\lambda_2(1-Y)+\lambda_1{}'(1-R)+\lambda_2{}'(1-S)+\mu_1\left(1-\frac{p_{12}Y}{X}-\frac{p_{13}Z}{X}\right)+\mu_2\left(1-\frac{p_{21}X}{Y}-\frac{p_{23}Z}{Y}\right)\\+\mu_3(1-\frac{1}{Z}-\frac{p_{31}X}{Z}-\frac{p_{32}Y}{Z}-\frac{p'_{31}R}{Z}-\frac{p'_{32}S}{Z})+\mu_1{}'(1-\frac{Z}{R})+\mu_2{}'(1-\frac{Z}{S})\end{bmatrix}}$$

(35)

Also F(1,1,1,1,1)=1 ,Being the total probability.

On taking X =1 as Y,Z,R,S →1 , F(X,Y,Z,R,S) is of $\frac{0}{0}$ indeterminate form.

Now, on differentiating using L- Hospital Rule (8) separately w.r.t X,Y,Z,R,S we have

$$1=\frac{\mu_1(p_{12}+p_{13})F_1+\mu_2(-p_{21})F_2+\mu_3(-p_{31})F_3}{-\lambda_1+\mu_1(p_{12}+p_{13})F_1+\mu_2(-p_{21})F_2+\mu_3(-p_{31})F_3}$$

$$\Rightarrow -\lambda_1+\mu_1(p_{12}+p_{13})F_1+\mu_2(-p_{21})F_2+\mu_3(-p_{31})F_3=\mu_1F_1-\mu_2p_{21}F_2-\mu_3p_{31}F_3$$

$$\Rightarrow -\lambda_1+\mu_1F_1+\mu_2(-p_{21})F_2+\mu_3(-p_{31})F_3=\mu_1F_1-\mu_2p_{21}F_2-\mu_3p_{31}F_3$$

By taking Y=1 and X,Z,R,S →1 we have

$$-\lambda_2 - \mu_1 p_{12} + \mu_2 + \mu_3 p_{32} = -\mu_1 p_{12} F_1 + \mu_2 F_2 + \mu_3 p_{32} F_3$$

By taking Z=1 and X,Y,R,S →1 we have

$$-\mu_1 p_{13} - \mu_2 p_{23} - \mu_1' - \mu_2' + 2\mu_3 = -\mu_1 p_{13} F_1 - \mu_2 p_{23} F_2 - \mu_1' F_4 - \mu_2' F_5 + 2\mu_3 F_3$$

By taking R=1 and X,Y,Z, S →1 we have

$$-\lambda_1' - \mu_3 p_{31}' + \mu_1' = -\mu_3 p_{31}' F_3 + \mu_1' F_4$$

By taking S=1 and X,Y,Z,R →1 We have

$$-\lambda_2' - \mu_3 p_{32}' + \mu_2' = -\mu_3 p_{32}' F_3 + \mu_1' F_5$$

By solving these equations we get,

$$(1 - F_1) = \frac{(\lambda_1 + \lambda_2 p_{21}) + (p_{31} + p_{32} p_{21})(\lambda_1 + \lambda_2 + \lambda_1' + \lambda_2')}{\mu_1 (1 + p_{12} p_{21})}$$

$$(1 - F_2) = \frac{(\lambda_1 p_{12} + \lambda_2) + (p_{31} p_{12} + p_{32})(\lambda_1 + \lambda_2 + \lambda_1' + \lambda_2')}{\mu_2 (1 + p_{12} p_{21})}$$

$$(1 - F_3) = \frac{(\lambda_1 + \lambda_2 + \lambda_1' + \lambda_2')}{\mu_3}$$

$$(1 - F_4) = \frac{\lambda_1' + p_{31}'(\lambda_1 + \lambda_2 + \lambda_1' + \lambda_2')}{\mu_1'}$$

$$(1 - F_5) = \frac{\lambda_2' + p_{32}'(\lambda_1 + \lambda_2 + \lambda_1' + \lambda_2')}{\mu_2'}$$

On using the values of F_1, F_2, F_3, F_4, F_5 the joint probability is given by

$$P_{n_1, n_2, n_3, n_4, n_5} = \rho_1^{n_1} \rho_2^{n_2} \rho_3^{n_3} \rho_4^{n_4} \rho_5^{n_5} (1 - \rho_1)(1 - \rho_2)(1 - \rho_3)(1 - \rho_4)(1 - \rho_5)$$

Where $\rho_1 = 1 - F_1$, $\rho_2 = 1 - F_2$, $\rho_3 = 1 - F_3$, $\rho_4 = 1 - F_4$, $\rho_5 = 1 - F_5$

5. MEAN QUEUE LENGTH

Average number of the customer (L)

$$= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \sum_{n_4=0}^{\infty} \sum_{n_5=0}^{\infty} (n_1 + n_2 + n_3 + n_4 + n_5) P_{n_1, n_2, n_3, n_4, n_5}$$

$$= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \sum_{n_4=0}^{\infty} \sum_{n_5=0}^{\infty} n_1 P_{n_1, n_2, n_3, n_4, n_5} + \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \sum_{n_4=0}^{\infty} \sum_{n_5=0}^{\infty} n_2 P_{n_1, n_2, n_3, n_4, n_5} + \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \sum_{n_4=0}^{\infty} \sum_{n_5=0}^{\infty} n_3 P_{n_1, n_2, n_3, n_4, n_5} + \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \sum_{n_4=0}^{\infty} \sum_{n_5=0}^{\infty} n_4 P_{n_1, n_2, n_3, n_4, n_5} + \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \sum_{n_4=0}^{\infty} \sum_{n_5=0}^{\infty} n_5 P_{n_1, n_2, n_3, n_4, n_5}$$

$L = L_1 + L_2 + L_3 + L_4 + L_5$

Where,

$L_1 = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\sum_{n_4=0}^{\infty}\sum_{n_5=0}^{\infty} n_1 P_{n_1,n_2,n_3,n_4,n_5}$

$= (1-\rho_1)(1-\rho_2)(1-\rho_3)(1-\rho_4)(1-\rho_5)$
$\sum_{n_1=0}^{\infty} n_1\rho_1^{n_1}\sum_{n_2=0}^{\infty} n_2\rho_2^{n_2}\sum_{n_3=0}^{\infty} n_3\rho_3^{n_3}\sum_{n_4=0}^{\infty} n_4\rho_4^{n_4}\sum_{n_5=0}^{\infty} n_5\rho_5^{n_5}$

$L_1 = \frac{\rho_1}{(1-\rho_1)}$

Similarly,

$L_2 = \frac{\rho_2}{(1-\rho_2)}$, $L_3 = \frac{\rho_3}{(1-\rho_3)}$, $L_4 = \frac{\rho_4}{(1-\rho_4)}$, $L_5 = \frac{\rho_5}{(1-\rho_5)}$

5.1 Variance of Queue:

$V(n_1+n_2+n_3+n_4+n_5) =$
$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\sum_{n_4=0}^{\infty}\sum_{n_5=0}^{\infty}(n_1+n_2+n_3+n_4+n_5)^2 P_{n_1,n_2,n_3,n_4,n_5} - L^2$

$=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\sum_{n_4=0}^{\infty}\sum_{n_5=0}^{\infty}(n_1)^2 P_{n_1,n_2,n_3,n_4,n_5} +$
$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\sum_{n_4=0}^{\infty}\sum_{n_5=0}^{\infty}(n_2)^2 P_{n_1,n_2,n_3,n_4,n_5} +$
$\cdots\ldots\ldots\ldots\ldots.. + \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\sum_{n_4=0}^{\infty}\sum_{n_5=0}^{\infty}(n_5)^2 P_{n_1,n_2,n_3,n_4,n_5} +$
$2\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\sum_{n_4=0}^{\infty}\sum_{n_5=0}^{\infty} n_1 n_2 P_{n_1,n_2,n_3,n_4,n_5} +$
$\cdots\ldots\ldots\ldots. + 2\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\sum_{n_4=0}^{\infty}\sum_{n_5=0}^{\infty} n_4 n_5 P_{n_1,n_2,n_3,n_4,n_5} - L^2$

$V = \frac{\rho_1}{(1-\rho_1)^2} + \frac{\rho_2}{(1-\rho_2)^2} + \frac{\rho_3}{(1-\rho_3)^2} + \frac{\rho_4}{(1-\rho_4)^2} + \frac{\rho_5}{(1-\rho_5)^2}$

5.2 Average waiting time for customer (W) = $\frac{L_Q}{\lambda}$

6. NUMERICAL ILLUSTRATION:

Give customers coming to three servers out of which one server consist two biserial channels and Other consists of three parallel service channels and furthers these two service channels are linked with common server. The number of customers, mean service rate, mean arrival rate and associated probabilities are given as follows:

S. No.	No. of Customers	Mean Service Rate	Mean arrival Rate	Probabilities
1	n_1=5	μ_1=12	λ_1= 2	p_{12}=0.4, p_{13}=0.6
2	n_2=6	μ_2=10	λ_2=4	p_{21}= 0.5, p_{23}= 0.5
3	n_3=8	μ_3=13	λ_1'=2	p_{31}=0.4, p_{31}'=0.3, p_{32}'=0.1 p_{32}= 0.2
4	n_4=3	μ_4=9	λ_2'=3	p_{12}'=0.3, p_{13}'=0.7
5	n_5=4	μ_5=8		p_{21}'= 0.8, p_{23}'= 0.2

Find the joint probability, mean queue length, variance of queue and average waiting time for customer.

Solution: - We have

$$\rho_1 = \frac{(\lambda_1+\lambda_2 p_{21})+(p_{31}+p_{32}p_{21})(\lambda_1+\lambda_2+\lambda_1'+\lambda_2')}{\mu_1(1+p_{12}p_{21})}$$

$$= \frac{(2+4\times0.5)+(0.4+0.1\times0.5)(2+4+2+3)}{(1-0.4\times0.5)12} = \frac{8.95}{9.6} = 0.93$$

$$\rho_2 = \frac{(\lambda_1 p_{12}+\lambda_2)+(p_{31}p_{12}+p_{32})(\lambda_1+\lambda_2+\lambda_1'+\lambda_2')}{\mu_2(1+p_{12}p_{21})}$$

$$= \frac{(2\times0.4+4)+(0.4\times0.4+0.1)(2+4+2+3)}{(1-0.4\times0.5)10} = \frac{7.66}{8} = 0.95$$

$$\rho_3 = \frac{(\lambda_1+\lambda_2+\lambda_1'+\lambda_2')}{\mu_3} = \frac{(2+4+2+3)}{13} = \frac{11}{13} = 0.84$$

$$\rho_4 = \frac{\lambda_1'+p_{31}'(\lambda_1+\lambda_2+\lambda_1'+\lambda_2')}{\mu_1'}$$

$$= \frac{2+0.3(2+4+2+3)}{9} = \frac{5.3}{9} = 0.58$$

$$\rho_5 = \frac{\lambda_2'+p_{32}'(\lambda_1+\lambda_2+\lambda_1'+\lambda_2')}{\mu_2'} = \frac{3+0.1(2+4+2+3)}{8} = \frac{4.1}{8} = 0.51$$

The joint probability is

$$P_{n_1,n_2,n_3,n_4,n_5} = \rho_1^{n_1}\rho_2^{n_2}\rho_3^{n_3}\rho_4^{n_4}\rho_5^{n_5}(1-\rho_1)(1-\rho_2)(1-\rho_3)(1-\rho_4)(1-\rho_5)$$

$$= (0.93)^6(0.95)^6(0.84)^8(0.58)^3(0.51)^4(0.07)(0.05)(0.16)(0.42)(0.49)$$

$$= 1.518\times10^{-7}$$

The Mean queue length (Average no. of customers)

$$L=\frac{\rho_1}{(1-\rho_1)^2}+\frac{\rho_2}{(1-\rho_2)}+\frac{\rho_3}{(1-\rho_3)}+\frac{\rho_4}{(1-\rho_4)}+\frac{\rho_5}{(1-\rho_5)}$$

$$=\frac{0.93}{0.07}+\frac{0.95}{0.05}+\frac{0.84}{0.16}+\frac{0.58}{0.42}+\frac{0.51}{0.49}$$

$$=39.95$$

Variance of queue

$$V=\frac{\rho_1}{(1-\rho_1)^2}+\frac{\rho_2}{(1-\rho_2)^2}+\frac{\rho_3}{(1-\rho_3)^2}+\frac{\rho_4}{(1-\rho_4)^2}+\frac{\rho_5}{(1-\rho_5)^2}$$

$$=\frac{0.93}{(0.07)^2}+\frac{0.95}{(0.05)^2}+\frac{0.84}{(0.16)^2}+\frac{0.58}{(0.42)^2}+\frac{0.51}{(0.49)^2}$$

$$=607.92$$

Average waiting time for customer

$$W=\frac{L}{\lambda}=\frac{39.95}{11}=3.631$$

7. CONCLUSION

Using the above relation we can find mean queue length and also variance in order to determine the fluctuation in queue length. Other decision making parameters are determine by using different formulae. If bi-series concept is not taken in first system then the work resembles with the work of T.P. Singh, Kusum etal. (2010). Further, if feedback concept is not taken in account then it resembles with the work of Deepak Gupta etel. (2010).

7.1 Further Research/ Extension of model: The model can be extended to a bitandem queue model if subsystem S_2 also contains two biserial service channels S_{21} & S_{22}. The customers getting service from S_{21} either go to S_{22} or S_3 with probabilities p_{12}' or p_{13}' such that $p_{12}'+p_{13}'=1$also servers getting service at S_{22} either go to S_{21} or S_3 with probabilities p_{21}' or p_{23}' such that $p_{21}'+p_{23}'=1$.

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