The number of Smallest parts of j^{th} *overpartitions* of n

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Abstract:

Sylvie Corteel and Jeremy Lovejoy [7] defined *overpartitions* and George E Andrews derived formula for the number of smallest parts of *partitions* of a positive integer *n*. In this paper the concepts of j^{th}*overpartition and* $r-j^{th}$*overpartition* of n are introduced and a formula for the number of smallest parts of the j^{th}*overpartition* is derived by using $r-j^{th}$*overpartitions* of n.

Keywords: *partition*, j^{th}*overpartition*, $r-j^{th}$*overpartition*, smallest parts of the *partition* and $r-j^{th}$*overpartition* of positive integer *n*.
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Introduction:

The author derived in his thesis [4] relationships between the i^{th} smallest parts and greatest parts for a *partitions* of n as well as *overpartitions.* The concept of *overpartition* of *n* [5],[6] may extended in a natural way to j^{th}*overpartition* for an integer *n*. It is natural to ask whether one can derive expressions for the number of smallest parts and that of greatest parts to this context.This requires the notion of *r - j^{th} overpartition of n*.

We divide this paper into two sections. Section 1 contains definitions and notation . In section 2 the main results on $j^{th} overpartitions.$

Section 1:

1.1 Definition of $j^{th} overpartition.$

The j^{th} *overpartition* of *n* is defined as a non-increasing sequence of natural numbers whose sum is *n* in which first (equivalently, the final) occurrence of a number may be j times *overlined*. We denote the j^{th} *overpartition* of λ, by $\overline{\lambda}^{j}$ and the number of j^{th} *overpartitions* of *n* by $\overline{p(n)}^{j}$. The *overlined* parts form a *partition* into distinct parts.

1.2 Example of $j^{th} overpartition$, the 27 *second overpartitions* of 4 are

4, $\overline{4}$, $\overline{\overline{4}}$, 3+1, $\overline{3}+1$, $\overline{\overline{3}}+1$, $3+\overline{1}$, $\overline{3}+\overline{1}$, $\overline{\overline{3}}+\overline{1}$, $3+\overline{\overline{1}}$, $\overline{3}+\overline{\overline{1}}$, $\overline{\overline{3}}+\overline{\overline{1}}$, 2+2, $\overline{2}+2$, $\overline{\overline{2}}+2$, 2+1+1, $\overline{2}+1+1$, $\overline{\overline{2}}+1+1$, $2+\overline{1}+1$, $\overline{2}+\overline{1}+1$, $\overline{\overline{2}}+\overline{1}+1$, $2+\overline{\overline{1}}+1$, $\overline{2}+\overline{\overline{1}}+1$, $\overline{\overline{2}}+\overline{\overline{1}}+1$, 1+1+1+1, $\overline{1}+1+1+1$, $\overline{\overline{1}}+1+1+1$

1.3 Definition of *r-j*th *overpartitions*:

The $j^{th} overpartition$ of n with *r* parts is called $r - j^{th} overpartition\, of\ n$.

1.4 Example of $r - j^{th} overpartition$, the 9, 3-*second overpartitions* of 4 are

2+1+1, $\overline{2}+1+1$, $\overline{\overline{2}}+1+1$, $2+\overline{1}+1$, $\overline{2}+\overline{1}+1$, $\overline{\overline{2}}+\overline{1}+1$, $2+\overline{\overline{1}}+1$, $\overline{2}+\overline{\overline{1}}+1$ and $\overline{\overline{2}}+\overline{\overline{1}}+1$

Let $\overline{\xi(n)}^{j}$ denote the set of all j^{th} *overpartitions* of n and $\overline{p(n)}^{j}$ the cardinality of $\overline{\xi(n)}^{j}$ for $n \in N$ and $\overline{p(0)}^{j} = 1$. If $1 \le r \le n$ write $\overline{p_r(n)}^{j}$ for the number of j^{th} *overpartitions* of n each consisting of exactly r parts, i.e $r - j^{th}$ *overpartitions* of *n.*

If $r \le 0$ or $r \ge n$ we write $\overline{p_r(n)}^{j} = 0$. Let $\overline{p(k,n)}^{j}$ represent the number of j^{th} *overpartitions* of n using natural numbers atleast as large as k only.

Let $\overline{spt(n)}^{j}$ denote the number of smallest parts including repetitions in all

j^{th} *overpartitions* of n . For $i \geq 1$ let us adopt the following notation.

$$m_s\left(\overline{\lambda}^j\right) = \text{ number of smallest parts of } \overline{\lambda}^j .$$

$$\overline{spt(n)}^j = \sum_{\lambda \in \xi(n)} m_s\left(\overline{\lambda}^j\right)$$

1.5 Existing generating functions are given below.

Function	Generating function
$p_r(n)$	$\frac{q^r}{(q)_r}$
$p_r(n-k)$	$\frac{q^{r+k}}{(q)_r}$
number of divisors	$\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)}$
sum of divisors	$\sum_{n=1}^{\infty}\frac{n.q^n}{(1-q^n)}$ (1.5.1)

where $(q)_k = \prod_{n=1}^{k}(1-q^n)$ for $k > o$, $(q)_k = 1$ for $k = o$ and $(q)_k = 0$ for $k < o$.

Since $(a)_n = (a;q)_n = (1-a)(1-aq)(1-aq^2)\ldots(1-aq^{n-1})$ [1]

1.6: Derivation for $r - j^{th}$ *overpartitions*:

$$\overline{p_1(n)}^j = (j+1)p_1(n) = \frac{(j+1)q}{(1-q)} = \frac{q(-j,q)_1}{(q)_1}$$

$$\overline{p_2(n)}^j = (1+j)^2 p_2(n) - (1+j)jp_1\left(\left[\frac{n}{2}\right]\right) = \frac{(1+j)^2 q^2}{(1-q)(1-q^2)} - \frac{(1+j)jq^2}{(1-q^2)}$$

$$= \frac{(1+j)q^2}{(1-q^2)}\left\{\frac{(1+j)}{(1-q)} - j\right\} = \frac{q^2(1+j)(1+jq)}{(1-q)(1-q^2)} = \frac{q^2(-j,q)_2}{(q)_2}$$

By induction, we get

$$\overline{p_r(n)}^j = \frac{q^r(1+j)(1+jq)(1+jq^2)\dots(1+jq^{r-1})}{(1-q)(1-q^2)(1-q^3)\dots(1-q^r)} = \frac{q^r(-j,q)_r}{(q)_r}$$

and $\overline{p_r(n-a)}^j = \dfrac{q^{r+a}(-j,q)_r}{(q)_r}$ (1.6.1)

And also we observe that the generating function for the number of j^{th} *overpartitions* is

$$\sum_{n=0}^{\infty} \overline{p(n)}^j q^n = \prod_{n=1}^{\infty} \frac{1+jq^{n-1}}{1-q^n}$$

Section 2:

2.1 Theorem: $\overline{spt(n)}^j = \sum_{k=1}^{\infty}\sum_{t=1}^{\infty} \overline{p(k,n-tk)}^j + j\sum_{k=1}^{\infty}\sum_{t=1}^{\infty} \overline{p(k+1,n-tk)}^j + (j+1)d(n)$ (2.1.1)

Proof :[2] Let $n = (\lambda_1, \lambda_2, \dots, \lambda_r) = \left(\mu_1^{\alpha_1}, \mu_2^{\alpha_2}, \dots, \mu_{l-1}^{\alpha_{l-1}}, k^{\alpha_l}\right)$ be any $r-partition$ of n with *l* distinct parts. For corresponding to it there are $(j+1)^l$ times $r-j^{th}overpartitions$ of *n.* (2.1.2)

Case 1: [3] Let $r > \alpha_l = t$ that means $\lambda_{r-t} > k$

Subtract all $k's$, we get $n-tk = \left(\mu_1^{\alpha_1}, \mu_2^{\alpha_2}, \dots, \mu_{l-1}^{\alpha_{l-1}}\right)$

Hence $n-tk = \left(\mu_1^{\alpha_1}, \mu_2^{\alpha_2}, \dots, \mu_{l-1}^{\alpha_{l-1}}\right)$ is a $(r-t)-partition$ of $n-tk$ with l - 1 distinct parts and each part greater than $k+1$. For corresponding to it they are $(j+1)^{l-1}$ times $(r-t)-j^{th}overpartitions$ of $n-tk$. From (2.1.2) we know that the total number of $r-j^{th}overpartitions$ are $(j+1)^l$.

Now we get, $(j+1)$ times the number $\overline{p_{r-t}(k+1,n-tk)}^j$ of $r-j^{th}overpartitions$ from $r-partitions$ of *n* with exactly t smallest elements as *k*.

Case 2: Let $r > \alpha_l > t$ that means $\lambda_{r-t} = k$

Omit $k's$ from last t places, we get $n-tk = \left(\mu_1^{\alpha_1}, \mu_2^{\alpha_2}, \dots, \mu_{l-1}^{\alpha_{l-1}}, k^{\alpha_l - t}\right)$

Hence $n-tk=\left(\mu_1^{\alpha_1},\mu_2^{\alpha_2},...,\mu_{l-1}^{\alpha_{l-1}},k^{\alpha_l-t}\right)$ is a $(r-t)-partition$ of $n-tk$ with l distinct parts and the least part is k. For corresponding to it there are $(j+1)^l$ times of $r-j^{th}overpartitions$ of $n-tk$ with least part k.

Now we get the number $\overline{f_{r-t}(k,n-tk)}^j$ of $r-j^{th}overpartitions$ from a $r-partitions$ of n with more than *t* smallest elements as *k*.

Case 3: Let $r=\alpha_l=t$ that means all parts in the *partition* are equal. For each $r-partition$ with equal parts have $(j+1)$ times of $r-j^{th}overpartitions$ of *n*.

From cases (1), (2) and (3) we get $r-j^{th}overpartitions$ of *n* with *t* smallest parts as *k* is

$$\overline{f_{r-t}(k,n-tk)}^j+(j+1).\overline{p_{r-t}(k+1,n-tk)}^j+(j+1)\beta$$

$$\text{where } \beta=1 \text{ if } r\,|\,n \text{ and } \beta=0 \text{ otherwise}$$

$$=\overline{f_{r-t}(k,n-tk)}^j+\overline{p_{r-t}(k+1,n-tk)}^j+j.\overline{p_{r-t}(k+1,n-tk)}^j+(j+1)\beta$$

$$=\overline{p_{r-t}(k,n-tk)}^j+j.\overline{p_{r-t}(k+1,n-tk)}^j+(j+1)\beta$$

The number of *partitions* of *n* with equal parts is equal to the number of divisors of *n*. Since the number of divisors of *n* is $d(n)$. Then the number of $j^{th}overpartitions$ of *n* with all parts are equal is $(j+1)d(n)$.

From [6], the number of smallest parts in $j^{th}overpartitions$ of *n* is

$$\overline{spt(n)}^j=\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}\overline{p(k,n-tk)}^j+j\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}\overline{p(k+1,n-tk)}^j+(j+1)d(n)$$

2.2 Theorem: $\overline{p_r(k+1,n)}^j=\overline{p_r(n-kr)}^j$ (2.2.1)

Proof : Let $n=(\lambda_1,\lambda_2,...,\lambda_r),\lambda_i>k\forall i$ be any $r-j^{th}$ *overpartition* of n.

Subtracting each part by k, we get $n-kr=(\lambda_1-k,\lambda_2-k,...,\lambda_r-k)$

Hence $n-kr=(\lambda_1-k,\lambda_2-k,...,\lambda_r-k)$ is a $r-j^{th}$ *overpartition* of $n-kr$.

Therefore the number of $r-j^{th}$ *overpartitions* of n with parts greater than or equal to $k+1$ is

$\overline{p_r\left(n-kr\right)}^j$

2.3 Theorem: $\sum_{n=0}^{\infty}\overline{spt(n)}^j q^n = \frac{(-j,q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}\frac{(j+1)q^n}{(1-q^n)}\frac{(q)_{n-1}}{(-j,q)_{n+1}}$

Proof: From theorem (2.2.1), we have

$$\overline{spt(n)}^j = \sum_{r=1}^{\infty}\sum_{t=1}^{\infty}\overline{p(k,n-tk)}^j + j\sum_{r=1}^{\infty}\sum_{t=1}^{\infty}\overline{p(k+1,n-tk)}^j + (j+1)d(n)$$

Replace $k+1$ by k, n by $n-tk$ for first part and n by $n-tk$ for second part in (2.2.1)

$$=\sum_{t=1}^{\infty}\sum_{r=1}^{\infty}\sum_{n=1}^{\infty}\overline{p_r\left(n-tk-r(k-1)\right)}^j + j\sum_{t=1}^{\infty}\sum_{r=1}^{\infty}\sum_{n=1}^{\infty}\overline{p_r\left(n-tk-rk\right)}^j + (j+1)d(n)$$

Where $d(n)$ is the number of positive divisors of n.

From (1.6.1)

$$=\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}\sum_{r=1}^{\infty}\frac{q^{r+tk+r(k-1)}(-j,q)_r}{(q)_r} + j\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}\sum_{r=1}^{\infty}\frac{q^{r+tk+rk}(-j,q)_r}{(q)_r} + \sum_{k=1}^{\infty}\frac{(j+1)q^k}{1-q^k}$$

$$=\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}\sum_{r=1}^{\infty}\frac{q^{tk+rk}(-j,q)_r}{(q)_r} + j\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}\sum_{r=1}^{\infty}\frac{q^{r+tk+rk}(-j,q)_r}{(q)_r} + \sum_{k=1}^{\infty}\frac{(j+1)q^k}{1-q^k}$$

$$=\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}q^{tk}\left[\sum_{r=1}^{\infty}\frac{(q^k)^r(-j,q)_r}{(q)_r}\right] + j\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}q^{tk}\left[\sum_{r=1}^{\infty}\frac{q^r(q^k)^r(-j,q)_r}{(q)_r}\right] + \sum_{k=1}^{\infty}\frac{(j+1)q^k}{1-q^k}$$

$$=\sum_{k=1}^{\infty}\frac{q^k}{(1-q^k)}\left[\left(1+\sum_{r=1}^{\infty}\frac{(q^k)^r(-j,q)_r}{(q)_r}\right)-1\right]$$

$$+j\sum_{k=1}^{\infty}\frac{q^k}{(1-q^k)}\left[\left(1+\sum_{r=1}^{\infty}\frac{(q^{k+1})^r(-j,q)_r}{(q)_r}\right)-1\right] + \sum_{r=1}^{\infty}\frac{(j+1)q^k}{1-q^k}$$

$$=\sum_{k=1}^{\infty}\frac{q^k}{(1-q^k)}\left(1+\sum_{r=1}^{\infty}\frac{(q^k)^r(-j,q)_r}{(q)_r}\right) + j\sum_{k=1}^{\infty}\frac{q^k}{(1-q^k)}\left(1+\sum_{r=1}^{\infty}\frac{(q^{k+1})^r(-j,q)_r}{(q)_r}\right)$$

$$=\sum_{k=1}^{\infty}\frac{q^k}{\left(1-q^k\right)}\prod_{r=0}^{\infty}\left(\frac{1+jq^rq^k}{1-q^rq^k}\right)+j\sum_{k=1}^{\infty}\frac{q^k}{\left(1-q^k\right)}\prod_{r=0}^{\infty}\left(\frac{1+jq^rq^{k+1}}{1-q^rq^{k+1}}\right)\qquad \text{from [1]}$$

$$=\sum_{k=1}^{\infty}\frac{q^k}{\left(1-q^k\right)}\prod_{r=0}^{\infty}\left(\frac{1+jq^{r+k}}{1-q^{r+k}}\right)+j\sum_{k=1}^{\infty}\frac{q^k}{\left(1-q^k\right)}\prod_{r=0}^{\infty}\left(\frac{1+jq^{r+k+1}}{1-q^{r+k+1}}\right)$$

$$=\frac{(-j,q)_\infty}{(q)_\infty}\sum_{k=1}^{\infty}\frac{q^k}{\left(1-q^k\right)}\frac{(q)_{k-1}}{(-j,q)_k}+j\frac{(-j,q)_\infty}{(q)_\infty}\sum_{k=1}^{\infty}\frac{q^k}{\left(1-q^k\right)}\frac{(q)_k}{(-j,q)_{k+1}}$$

$$=\frac{(-j,q)_\infty}{(q)_\infty}\sum_{k=1}^{\infty}\frac{q^k}{\left(1-q^k\right)}\frac{(q)_{k-1}}{(-j,q)_k}\left[1+j\frac{\left(1-q^k\right)}{\left(1+jq^k\right)}\right]$$

$$=\frac{(-j,q)_\infty}{(q)_\infty}\sum_{k=1}^{\infty}\frac{(j+1)q^k}{\left(1-q^k\right)}\frac{(q)_{k-1}}{(-j,q)_{k+1}}$$

$$=\frac{(-j,q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}\frac{(j+1)q^n}{\left(1-q^n\right)}\frac{(q)_{n-1}}{(-j,q)_{n+1}}$$

2.4 Corollary: The generating function for the number $\overline{A_c(n)}^{\,j}$ of smallest parts of the j^{th} *overpartitions* of n which are multiples of c is

$$\sum_{n=0}^{\infty}\overline{A_c(n)q^n}^{\,j}=\frac{(-j,q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}\frac{(j+1)q^{cn}}{\left(1-q^{cn}\right)}\frac{(q)_{cn-1}}{(-j,q)_{cn+1}}$$

2.5 Corollary: The generating function for the sum of smallest parts of the second j^{th} *overpartitions* of n is

$$\sum_{n=0}^{\infty}\overline{sum\,spt(n)q^n}^{\,j}=\frac{(-j,q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}\frac{(j+1)nq^n}{\left(1-q^n\right)}\frac{(q)_{n-1}}{(-j,q)_{n+1}}$$

Proof: The generating function for the sum of smallest parts of the second j^{th} *overpartitions* of a positive integer n is

$$\overline{spt(n)}^{j}=\sum_{r=1}^{\infty}\sum_{t=1}^{\infty}k\ \overline{p(k,n-tk)}^{j}+j\sum_{r=1}^{\infty}\sum_{t=1}^{\infty}k\ \overline{p(k+1,n-tk)}^{j}+(j+1)d(n)$$

where

$$=\sum_{t=1}^{\infty}\sum_{r=1}^{\infty}\sum_{n=1}^{\infty}k\ \overline{p_r\left(n-tk-r(k-1)\right)}^{j}+j\sum_{t=1}^{\infty}\sum_{r=1}^{\infty}\sum_{n=1}^{\infty}k\ \overline{p_r\left(n-tk-rk\right)}^{j}+(j+1)d(n)$$

$d(n)$ is the number of positive divisors of n.

From (1.6.1)

$$=\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}\sum_{r=1}^{\infty}\frac{kq^{r+tk+r(k-1)}(-j,q)_r}{(q)_r}+j\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}\sum_{r=1}^{\infty}\frac{kq^{r+tk+rk}(-j,q)_r}{(q)_r}+\sum_{k=1}^{\infty}\frac{(j+1)kq^k}{1-q^k}$$

$$=\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}\sum_{r=1}^{\infty}\frac{kq^{tk+rk}(-j,q)_r}{(q)_r}+j\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}\sum_{r=1}^{\infty}\frac{kq^{r+tk+rk}(-j,q)_r}{(q)_r}+\sum_{k=1}^{\infty}\frac{(j+1)kq^k}{1-q^k}$$

$$=\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}kq^{tk}\left[\sum_{r=1}^{\infty}\frac{(q^k)^r(-j,q)_r}{(q)_r}\right]+j\sum_{k=1}^{\infty}\sum_{t=1}^{\infty}kq^{tk}\left[\sum_{r=1}^{\infty}\frac{q^r(q^k)^r(-j,q)_r}{(q)_r}\right]+\sum_{k=1}^{\infty}\frac{(j+1)kq^k}{1-q^k}$$

$$=\sum_{k=1}^{\infty}\frac{kq^k}{(1-q^k)}\left[\left(1+\sum_{r=1}^{\infty}\frac{(q^k)^r(-j,q)_r}{(q)_r}\right)-1\right]$$

$$+j\sum_{k=1}^{\infty}\frac{kq^k}{(1-q^k)}\left[\left(1+\sum_{r=1}^{\infty}\frac{(q^{k+1})^r(-j,q)_r}{(q)_r}\right)-1\right]+\sum_{r=1}^{\infty}\frac{(j+1)kq^k}{1-q^k}$$

$$=\sum_{k=1}^{\infty}\frac{kq^k}{(1-q^k)}\prod_{r=0}^{\infty}\left(\frac{1+jq^rq^k}{1-q^rq^k}\right)+j\sum_{k=1}^{\infty}\frac{kq^k}{(1-q^k)}\prod_{r=0}^{\infty}\left(\frac{1+jq^rq^{k+1}}{1-q^rq^{k+1}}\right)$$

$$=\sum_{k=1}^{\infty}\frac{kq^k}{(1-q^k)}\prod_{r=0}^{\infty}\left(\frac{1+jq^{r+k}}{1-q^{r+k}}\right)+j\sum_{k=1}^{\infty}\frac{kq^k}{(1-q^k)}\prod_{r=0}^{\infty}\left(\frac{1+jq^{r+k+1}}{1-q^{r+k+1}}\right)$$

$$=\frac{(-j,q)_\infty}{(q)_\infty}\sum_{k=1}^{\infty}\frac{kq^k}{(1-q^k)}\frac{(q)_{k-1}}{(-j,q)_k}+j\frac{(-j,q)_\infty}{(q)_\infty}\sum_{k=1}^{\infty}\frac{kq^k}{(1-q^k)}\frac{(q)_k}{(-j,q)_{k+1}}$$

$$=\frac{(-j,q)_\infty}{(q)_\infty}\sum_{k=1}^{\infty}\frac{kq^k}{(1-q^k)}\frac{(q)_{k-1}}{(-j,q)_k}\left[1+j\frac{(1-q^k)}{(1+jq^k)}\right]$$

$$=\frac{(-j,q)_\infty}{(q)_\infty}\sum_{k=1}^{\infty}\frac{(j+1)kq^k}{(1-q^k)}\frac{(q)_{k-1}}{(-j,q)_{k+1}}$$

$$=\frac{(-j,q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}\frac{(j+1)nq^n}{(1-q^n)}\frac{(q)_{n-1}}{(-j,q)_{n+1}}$$

$$\sum_{n=0}^{\infty}\overline{sum\,spt(n)q^n}^{\,j}=\frac{(-j,q)_\infty}{(q)_\infty}\sum_{n=1}^{\infty}\frac{(j+1)nq^n}{(1-q^n)}\frac{(q)_{n-1}}{(-j,q)_{n+1}}$$

$$=\sum_{g_1=1}^{\infty}\left[\frac{g_1.q^{g_1}(-j,q)_{g_1}}{(q)_{g_1}}-\frac{(j+1)q^{g_1}}{(1-q^{g_1})}\sum_{g_2=1}^{g_1-1}\frac{g_2.q^{g_2}(-j,q)_{g_2}}{(q)_{g_2}}\right]$$

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References

[1] Andrews, G. E. (1998), The Theory of Partitions, *Cambridge University Press*, Cambridge. MR **99c:**11126.

[2] HanumaReddy.K. (2009), A Note on *r – partitions* of *n* in which the least part is *k*, International Journal of Computational Mathematical Ideas, **2,**1,pp. 6-12.

[3] HanumaReddy.K. (2010), A Note on *partitions* , International Journal of Mathematical Sciences, **9,** 3-4, pp. 313-322.

[4] HanumaReddy.K. (2011), Thesis, A Note on *r – partitions*, Acharya Nagarjuna University, Andhra Pradesh, India.

[5] Ramabhadra Sarma.I, Hanuma Reddy.K, S.Rao Gunakala and D.M.G. Comissiong, (2011), Relation between Smallest and Greatest Parts of the Partitions of *n* ,Journal of Mathematics Research,3,4, pp. 133-140.

[6] Ramabhadra Sarma.I, Hanuma Reddy.K, S.Rao Gunakala and D.M.G. Comissiong,(2011), Relation between Smallest and Greatest Parts of the Overpartitions of *n* , International Electronic Journal of Pure and Applied Mathematics , 3,3, pp. 195-205.

[7] Sylvie Corteel and Jeremy Lovejoy, (2003), Overpartitions,Transactions of the *American Mathematical Society*, **356,** 4, pp. 1623-1635.