International Journal of Research in Engineering and Applied Sciences (IJREAS)
Available online at http://euroasiapub.org/journals.php
Vol. 6 Issue 12, December - 2016, pp. 20~28
ISSN (O): 2249-3905, ISSN(P) : 2349-6525 | Impact Factor: 6.573 | Thomson Reuters ID: L-5236-2015

RATIONAL INEQUALITIES AND COMMON FIXED POINT THEOREMS GOVERNED BY INTIMATE MAPPINGS IN MULTIPLICATIVE METRIC SPACES

Arti Mishra

Manav Rachna International University, Faridabad-121004, Haryana, INDIA

Abstract- This paper is in the sequence of papers which satisfies the common fixed point theorems, but the aim of this paper is to prove the common fixed point result using a noble concept of rational contractive mapping using four maps, in which the pair of the maps are satisfying the mapping of type (A), also the pairs of maps are assumed to satisfy the intimate mapping in the setup of multiplicative metric spaces.

AMS Subject Classification: 47H10, 54H25

Key words- rational contractive map, compatible Mappings of type (A) and intimate mappings, complete multiplicative metric space,

1. INTRODUCTION

Özavsar and Cevikel [1] investigated multiplicative metric spaces by defining its topological properties also the concept of multiplicative contraction and some fixed point results had proved by the said researchers. Intimate mappings have been introduced in 2001 by Sahu et al. [2]. It is the fact that Intimate mappings are nothing but the extension of the Compatible mappings of type (A) and this very important type of mapping was introduced by Kang et al. [3]. There is a difference between the type of mappings that is the mappings like Compatible mappings of type (A), weakly Compatible mappings and weakly Compatible mappings of type (A) and intimate mappings are the fact of commutativity at coincidence point. Coincidence point is the point at which Compatible mappings of type (A), weakly Compatible mappings and weakly Compatible mappings of type (A) commutes. But Intimate mappings are different in this way because it is not necessary for intimate mapping to commute at a point of coincidence.

We are going to use four self-maps and the rational functions to prove the unique common fixed point theorem. The pairs of these maps are chosen in such a way that each pair satisfies the intimate mapping as well as the compatible mapping of type (A). We are going to use some basic notations like letters $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{N}$ which represents real numbers, the set of all positive real numbers and the set of all natural numbers in general the concept of multiplicative. Metric space was introduced by Basihorov et. al.[4] and this new concept of multiplicative metric space was used in biomedical image analysis to prove the fundamental theorem of multiplicative calculus. It was also proved that the ordinary differential equations were less suitable than the multiplicative differential equation.

Now , some important and necessary definitions and results in multiplicative metric space in sequel are as follows:

Definition 1.1.[4] Let X be a nonempty set. Multiplicative metric [5] is a mapping

$d : X \times X \to \mathbb{R}_+$ satisfying the following conditions:

(m1) $d(x, y) \geq 1$ for all $x, y \in X$ and $d(x, y) = 1$ if and only if $x = y$,

(m2) $d(x, y) = d(y, x)$ for all $x, y \in X$,

(m3) $d(x, z) \leq d(x, y) \cdot d(y, z)$ for all $x, y, z \in X$ (multiplicative triangle inequality).

One of the example of complete multiplicative metric space is $\mathbb{R}_+$ with respect to multiplicative metric..

Definition 1.2. ([1]) (Multiplicative interior point): Let (X, d) be a multiplicative metric space and $A \subset X$. Then we call $x \in A$ a multiplicative interior point of A if there exists an $\varepsilon > 1$ such that $B_\varepsilon(x) \subset A$. The collection of all interior points of A is called multiplicative interior of A and denoted by int(A).

Definition 1.3. ([1]) (Multiplicative open set) Let (X, d) be a multiplicative metric space and $A \subset X$. If every point of A is a multiplicative interior point of A, i.e., A = int(A), then A is called a multiplicative open set.

Definition 1.4. ([1]) Let (X, d) be a multiplicative metric space. A point $x \in X$ is said to be multiplicative limit point of $S \subset X$ if and only if $(B_\varepsilon(x) \setminus \{x\}) \cap S \neq \emptyset$ for every $\varepsilon > 1$. The set of all multiplicative limit points of the set S is denoted by S'.

Definition 1.5. ([1]) Let (X, d) be a multiplicative metric space. We call a set $S \subset X$ multiplicative closed in (X, d) if S contains all of its multiplicative limit points.

The following propositions can be easily proven by the definition of multiplicative closed set:

Definition 1.6. ([1]) (Multiplicative continuity) Let (X, d_X) and (Y, d_Y) be two multiplicative metric spaces and $f : X \to Y$ be a function. If f holds the requirement that, for every $\varepsilon > 1$, there exists $\delta > 1$ such that $f(B_\delta(x)) \subset B_\varepsilon(f(x))$, then we call f multiplicative continuous at $x \in X$.

Definition 1.7. ([5]) (Multiplicative convergence): Let (X, d) be a multiplicative metric space, $\{x_n\}$ be a sequence in X and $x \in X$. If for every multiplicative open ball $B_\varepsilon(x)$, there exists a natural number $\mathbb{N}$ such that $n \geq \mathbb{N} \Rightarrow x_n \in B_\varepsilon(x)$, then the sequence $\{x_n\}$ is said to be multiplicative convergent to x, denoted by $x_n \to x$ $(n \to \infty)$.

Lemma 1.8. ([5]) Let (X, d) be a multiplicative metric space, $\{x_n\}$ be a sequence in X and $x \in X$. Then $x_n \to x$ $(n \to \infty)$ if and only if $d(x_n, x) \to 1$ $(n \to \infty)$.

$\frac{1}{\varepsilon} < d(x_n, x) < 1 \cdot \varepsilon$ for all $n \geq \mathbb{N}$.

Lemma 1.9. ([5]) Let (X, d) be a multiplicative metric space, $\{x_n\}$ be a sequence in X. If the sequence $\{x_n\}$ is multiplicative convergent, then the multiplicative limit point is unique.

Theorem 1.10. ([5]) Let (X, d) be a multiplicative metric space and $\{x_n\}$ be a sequence in X. The sequence is multiplicative convergent, then it is a multiplicative Cauchy sequence.

Definition 1.11. ([5]) Let (X, d) be a multiplicative metric space and $A \subset X$. The set A is called multiplicative bounded if there exist $x \in X$ and $M > 1$ such that $A \subseteq B_M(x)$.

Lemma 1.12. ([1]) Let (X, d) be a multiplicative metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a multiplicative Cauchy sequence if and only if $d(x_n, x_m) \to 1$ $(m, n \to \infty)$.

Theorem 1.13. ([5]) Let (X, d) be a multiplicative metric space. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that $x_n \to x$, $y_n \to y$ $(n \to \infty)$, $x, y \in X$. Then

$d(x_n, y_n) \to d(x, y)$ $(n \to \infty)$.

Theorem 1.14. ([1]) Let $\{x_n\}$ be a multiplicative Cauchy sequence in a multiplicative metric space (X, d). If the sequence $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ such that

$x_{n_k} \to x \in X$ $(n_k \to \infty)$, then $\{x_n\} \to x$ $(n \to \infty)$.

Definition 1.15. ([5]) Let S and T be self-maps of multiplicative metric space a non-empty set X. then

i. Any point $x \in X$ is said to be fixed point of T if $Tx=x$.
ii. Any point $x \in X$ is said to be coincidence point of T and S if $Sx=Tx$ and we shall called $w=Sx=Tx$ that a point of coincidence of S and T.
iii. Any point $x \in X$ is said to be fixed point of T and S if $Sx=Tx=x$

Definition 1.16. ([5]) Let f and g be two mappings of a multiplicative metric space (X, d) into itself. Then f and g are said to be

(1) Compatible if

$$\lim_{n\to\infty} d(fgx_n, gfx_n) = 1$$

Whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$$

for some $t \in X$.

(2) Compatible of type (A) if

$\lim_{n\to\infty} d(fgx_n, ggx_n) = 1$ and $\lim_{n\to\infty} d(gfx_n, ffx_n) = 1$

Whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$$

for some $t \in X$.

Definition 1.17. ([3]) Let f and g be two mappings of a multiplicative metric Space (X, d) into itself. Then f and g are said to be

(1) g-intimate mappings if

$\alpha d(gfx_n, gx_n) \le \alpha d(ffx_n, fx_n)$,

Where α = lim sup or lim inf and $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$$

for some $t \in X$.

(2) f-intimate mappings if

$\alpha d(fgx_n, fx_n) \leq \alpha d(ggx_n, gx_n)$,

where α = limsup or liminf and $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$$

for some $t \in X$.

Proposition 1.18. ([3]) Let f and g be mappings of a multiplicative metric space (X, d) into itself. Assume that f and g are compatible of type (A). Then f and g are f-intimate and g-intimate.

Proposition 1.19. ([5]) Let f and g be mappings of a multiplicative metric space (X, d) into itself. Assume that f and g are g-intimate and $ft = gt = p \in X$. Then $d(gp, p) \leq d(fp, p)$.

Proof. Suppose that $x_n = t$ for all $n \geq 1$. So

$fx_n = gx_n \to ft = gt = p$.

Since f and g are g-intimate, we have

$d(gft, gt) = \lim_{n\to\infty} d(gfx_n, gx_n) \leq \lim_{n\to\infty} d(ffx_n, fx_n) = d(fft, ft)$.

This implies that, $d(gp, p) \leq d(fp, p)$.

1. Main Results

Now we prove a common fixed point theorem using the the above definitions and results.

Theorem 2.1. Let A, B, S and T be mappings of a complete multiplicative metric space (X, d) into itself satisfying the conditions

(2.1) $SX \subset BX$ and $TX \subset AX$

(2.2)
$$d(Sx,Ty) \leq \left\{ \max \left\{ \begin{array}{l} \frac{d(Ax,Ty)[d(Ax,Sx)+d(Ty,By)]}{d(Sx,Ty)+d(By,Ax)}, \\ \frac{d(Ax,By)[d(Ty,Sx)+d(Ax,Ty)]}{d(By,Ax)+d(Sx,By)}, \\ \frac{d(Ty,Sx)[d(By,Ax)+d(Sx,By)]}{d(Ty,Sx)+d(Ax,Ty)}, \\ \frac{d(Ty,Sx)[d(By,Ty)+d(By,Ax)]}{d(Ax,Sx)+d(Ty,Sx)} \end{array} \right\} \right\}^{\lambda}$$

for all $x, y \in X$, also $\lambda \in (0,1/2)$. Assume the following conditions are satisfied

(a) one of the mappings A, B, S and T is continuous
(b) Assume that the pairs A, S and B, T are weakly commuting
(c) Assume that the pair A, S and B, T is compatible of type (A).
(d) Assume that A(X) is complete and the pairs A, S is A-intimate and B, T is B-intimate.

Then A, B, S and T have unique common fixed point.

Proof. Let us consider any arbitrary point x_0 of X.
As given $SX \subset BX$. Hence there exist another point x_1 of X in such a way that,
$Sx_0 = Bx_1 = y_0$.
for the chosen x_1, there exist any x_2 of X, in such a way that
$Tx_1 = Ax_2 = y_1$.
Using the induction, there exist sequences $\{x_n\}$ and $\{y_n\}$ in X, so that
$Sx_{2n} = Bx_{2n+1} = y_{2n}$, $Tx_{2n+1} = Ax_{2n+2} = y_{2n+1}$,
using (2.2), we have

$$d(y_{2n}, y_{2n+1}) \leq d(Sx_{2n}, Tx_{2n+1})$$

$$\le \left\{\max\left\{\begin{array}{c}\frac{d(Ax_{2n},Tx_{2n+1})[d(Ax_{2n},Bx_{2n+1})+d(Tx_{2n+1},Sx_{2n})]}{d(Sx_{2n},Tx_{2n+1})+d(Bx_{2n+1},Ax_{2n})},\\ \frac{d(Ax_{2n},Bx_{2n+1})[d(Tx_{2n+1},Sx_{2n})+d(Ax_{2n},Tx_{2n+1})]}{d(Bx_{2n+1},Ax_{2n})+d(Sx_{2n},Bx_{2n+1})},\\ \frac{d(Tx_{2n+1},Sx_{2n})[d(Bx_{2n+1},Ax_{2n})+d(Sx_{2n},Bx_{2n+1})]}{d(Tx_{2n+1},Sx_{2n})+d(Ax_{2n},Tx_{2n+1})},\\ \frac{d(Tx_{2n+1},Sx_{2n})[d(Bx_{2n+1},Tx_{2n+1})+d(Bx_{2n+1},Ax_{2n})]}{d(Ax_{2n},Sx_{2n})+d(Tx_{2n+1},Sx_{2n})}\end{array}\right\}\right\}^{\lambda}$$

$$\le \left\{\max\left\{\begin{array}{c}\frac{d(y_{2n-1},y_{2n+1})[d(y_{2n-1},y_{2n})+d(y_{2n+1},y_{2n})]}{d(y_{2n},y_{2n+1})+d(y_{2n},y_{2n-1})},\\ \frac{d(y_{2n-1},y_{2n})[d(y_{2n+1},y_{2n})+d(y_{2n-1},y_{2n+1})]}{d(y_{2n},y_{2n-1})+d(y_{2n},y_{2n})},\\ \frac{d(y_{2n+1},y_{2n})[d(y_{2n},y_{2n-1})+d(y_{2n},y_{2n})]}{d(y_{2n+1},y_{2n})+d(y_{2n-1},y_{2n+1})},\\ \frac{d(y_{2n+1},y_{2n})[y_{2n},y_{2n+1})+d(y_{2n},y_{2n-1})]}{d(y_{2n-1},y_{2n})+d(y_{2n+1},y_{2n})}\end{array}\right\}\right\}^{\lambda}$$

$$\le \left\{\max\left\{\begin{array}{c}d(y_{2n-1},y_{2n+1}),d(y_{2n-1},y_{2n}).d(y_{2n+1},y_{2n}),\\ d(y_{2n+1},y_{2n}),d(y_{2n+1},y_{2n})\end{array}\right\}\right\}^{\lambda}$$

$d(y_{2n},y_{2n+1})$

$$\le \left\{\max\left\{\begin{array}{c}d(y_{2n-1},y_{2n+1}),d(y_{2n-1},y_{2n}).d(y_{2n+1},y_{2n}),\\ d(y_{2n+1},y_{2n}),1\end{array}\right\}\right\}^{\lambda}$$

$$d(y_{2n},y_{2n+1}) \le d^{\frac{\lambda}{1-\lambda}}(y_{2n-1},y_{2n})$$

$$d(y_{2n},y_{2n+1}) \le d^{h}(y_{2n-1},y_{2n}),$$ where $h=\frac{\lambda}{1-\lambda}$

In the same way,

$$d(y_{2n+1},y_{2n+2}) \le d(Tx_{2n+1},Sx_{2n+2})$$

$$= d(Sx_{2n+2},Tx_{2n+1})$$

$$\le \left\{\max\left\{\begin{array}{c}\frac{d(Ax_{2n+2},Tx_{2n+1})[d(Ax_{2n+2},Bx_{2n+2})+d(Tx_{2n+1},Bx_{2n})]}{d(Sx_{2n+2},Tx_{2n+1})+d(Bx_{2n+1},Ax_{2n+2})},\\ \frac{d(Ax_{2n+2},Bx_{2n+1})[d(Tx_{2n+1},Sx_{2n+2})+d(Ax_{2n+2},Tx_{2n+1})]}{d(Bx_{2n+1},Ax_{2n+2})+d(Sx_{2n+2},Bx_{2n+1})},\\ \frac{d(Tx_{2n+1},Sx_{2n+2})[d(Bx_{2n+1},Ax_{2n+2})+d(Sx_{2n+2},Bx_{2n+1})]}{d(Tx_{2n+1},Sx_{2n+2})+d(Ax_{2n+2},Tx_{2n+1})},\\ \frac{d(Tx_{2n+1},Sx_{2n+2})[d(Bx_{2n+1},Tx_{2n+1})+d(Bx_{2n+1},Ax_{2n+2})]}{d(Ax_{2n+2},Sx_{2n+2})+d(Tx_{2n+1},Sx_{2n+2})}\end{array}\right\}\right\}^{\lambda}$$

$$\le \left\{\max\left\{\begin{array}{c}\frac{d(y_{2n+1},y_{2n+1})[d(y_{2n+1},y_{2n+2})+d(y_{2n+1},y_{2n})]}{d(y_{2n+2},y_{2n+1})+d(y_{2n},y_{2n+1})},\\ \frac{d(y_{2n+1},y_{2n})[d(y_{2n+1},y_{2n+2})+d(y_{2n+1},y_{2n+1})]}{d(y_{2n},y_{2n+1})+d(y_{2n+2},y_{2n})},\\ \frac{d(y_{2n+1},y_{2n+2})[d(y_{2n},y_{2n+1})+d(y_{2n+2},y_{2n})]}{d(y_{2n+1},y_{2n+2})+d(y_{2n+1},y_{2n+1})},\\ \frac{d(y_{2n+1},y_{2n+2})[y_{2n},y_{2n+1})+d(y_{2n},y_{2n+1})]}{d(y_{2n+1},y_{2n+2})+d(y_{2n+1},y_{2n+2})}\end{array}\right\}\right\}^{\lambda}$$

(using (m1, m2, m3))

$$\le \left\{\max\left\{\begin{array}{c}1,1,d(y_{2n+1},y_{2n}).d(y_{2n},y_{2n+1}),\\ d(y_{2n+1},y_{2n})\end{array}\right\}\right\}^{\lambda}$$

$$d(y_{2n+1},y_{2n+2}) \le d^{\frac{\lambda}{1-\lambda}}(y_{2n},y_{2n+1})$$

$$d(y_{2n+1},y_{2n+2}) \le d^{h}(y_{2n},y_{2n+1})$$

(since, $h=\frac{\lambda}{1-\lambda}$)

Hence

$$d(y_n,y_{n+1}) \le d^{h^1}(y_{n-1},y_n) \le d^{h^2}(y_{n-1},y_n)$$
$$\le d^{h^3}(y_{n-1},y_n) \le \cdots$$
$$\le d^{h^n}(y_0,y_1)$$

Therefore,

for

All $m,n \in \mathbb{N}, n >$
$m,$ *using the triangular inequality of multiplicative metric space*

we have,

$$d(y_m, y_n) \leq d^{h^1}(y_m, y_{m-1}) \leq d^{h^2}(y_{m-1}, y_{m-2}) \leq d^{h^3}(y_{m-2}, y_{m-3}) \leq \cdots \leq d^{h^n}(y_{n+1}, y_n)$$

$$d(y_m, y_n) \leq d^{h^{m-1}}(y_1, y_0).\, d^{h^{m-2}}(y_1, y_0).\, d^{h^{m-2}}(y_1, y_0)\ldots .\, d^{h^n}(y_1, y_0)$$

$$d(y_m, y_n) \leq d^{\frac{h^n}{1-h}}(y_1, y_0).$$

This implies that $d(y_m, y_n)$ coverges to 1, as n, m approaches to ∞.

Therefore $\{y_n\}$is a multiplicative sequence sequence.

Since, we have mentioned that X is a complete multiplicative metric space and hence the convergent point belong to X.

Since, AX is complete there exists p ∈ AX such that

$y_{2n+1} = Tx_{2n+1} = Ax_{2n+2}$ approaches to p as n approaches to ∞.

Continuingly, there exist u ∈ X such that

(2.3) Au = p.

if $\{y_n\}$ is a cauchy sequence then $\{y_{2n+1}\}$ and $\{y_{2n}\}$ i.e. all subsequences of $\{y_n\}$ are also convergent. Therefore,

We have

$y_{2n} = Sx_{2n} = Bx_{2n+1}$ approaches to p as n approaches to ∞. i.e. $\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} Bx_{2n+1} = p$

Let's have an assertion, Su = p, Using the replacement as x=u and y = x_{2n+1} and using equation (2.2)

We have,

$$d(Su, Tx_{2n+1}) \leq \left\{\max\left\{\begin{array}{c}\frac{d(Au,Tx_{2n+1})[d(Au,Su)+d(Tx_{2n+1},Bx_{2n+1})]}{d(Su,Tx_{2n+1})+d(Bx_{2n+1},Au)},\\ \frac{d(Au,Bx_{2n+1})[d(Tx_{2n+1},Su)+d(Au,Tx_{2n+1})]}{d(Bx_{2n+1},Au)+d(Su,Bx_{2n+1})},\\ \frac{d(Tx_{2n+1},Su)[d(Bx_{2n+1},Au)+d(Su,Bx_{2n+1})]}{d(Tx_{2n+1},Su)+d(Au,Tx_{2n+1})},\\ \frac{d(Tx_{2n+1},Su)[d(Bx_{2n+1},Tx_{2n+1})+d(Bx_{2n+1},Au)]}{d(Au,Su)+d(Tx_{2n+1},Su)}\end{array}\right\}\right\}^{\lambda}$$

taking the limiting value of n as ∞, we have

$$d(Su, p) \leq \left\{\max\left\{\begin{array}{c}\frac{d(Au,p)[d(Au,Su)+d(p,p)]}{d(Su,p)+d(p,Au)}, \frac{d(Au,p)[d(p,Su)+d(Au,p)]}{d(p,Au)+d(Su,p)},\\ \frac{d(p,Su)[d(p,Au)+d(Su,p)]}{d(p,Su)+d(Au,p)},\\ \frac{d(p,Su)[d(p,p)+d(p,Au)]}{d(Au,Su)+d(p,Su)}\end{array}\right\}\right\}^{\lambda}$$

$$d(Su, p) \leq \left\{\max\left\{\begin{array}{c}1,1\\ d(p, Su),\\ 1\end{array}\right\}\right\}^{\lambda} \qquad \text{(using (2.3))}$$

which is a contradiction, since λ ∈(0,1/2). And hence, d(Su, p) = 1, i.e. Su = p.

(2.4) Su = Au = p.

Next we have assertion that p = Tv. Also p = Su ∈ SX ⊂ BX

Therefore there exist a point v ∈ X, in such a way that

(2.5) Bv = p.

By the substitution as x = u and y = v in inequality (2.2) we have

d(p, Tv) = d(Su, Tv)

$\leq$

$$\left\{\max\left\{\begin{array}{c}\frac{d(Au,Tv)[d(Au,Su)+d(Tv,Bv)]}{d(Su,Tv)+d(Bv,Au)},\\ \frac{d(Au,Bv)[d(Tv,Su)+d(Au,Tv)]}{d(Bv,Au)+d(Su,Bv)},\\ \frac{d(Tv,Su)[d(Bv,Au)+d(Su,Bv)]}{d(Tv,Su)+d(Au,Tv)},\\ \frac{d(Tv,Su)[d(Bv,Tv)+d(Bv,Au)]}{d(Au,Su)+d(Tv,Su)}\end{array}\right\}\right\}^{\lambda}$$

$=$

$$\left\{\max\left\{\begin{array}{c}\frac{d(p,Tv)[d(p,p)+d(Tv,p)]}{d(p,Tv)+d(p,p)},\\ \frac{d(p,p)[d(Tv,p)+d(p,Tv)]}{d(p,p)+d(p,p)},\\ \frac{d(Tv,p)[d(p,p)+d(p,p)]}{d(Tv,p)+d(p,Tv)},\\ \frac{d(Tv,p)[d(p,Tv)+d(p,p)]}{d(p,p)+d(Tv,p)}\end{array}\right\}\right\}^{\lambda}$$

(using(2.4) and (2.5))

$=\{\max\{(p,Tv), d(p,Tv), 1, d(Tp,p)\}\}^{\lambda}$

$d(p, Tv) = d^{\lambda}(p,Tv)$

this is a contradiction because $\lambda \in (0,1/2)$. And hence,

(2.6) Tv=p

Using 2.4 we have Su= Au = p and , it is given that A, S is A-intimate, by proposition 1.22.

we have

$d(Ap, p) \leq d(Sp, p)$.

let us consider that $Sp \neq p$ and using the condition (2.2), we get

$d(Sp, p)=d(Sp,Tv)$

$\leq$

$$\left\{\max\left\{\begin{array}{c}\frac{d(Au,Tp)[d(Au,Su)+d(Tp,Bp)]}{d(Su,Tp)+d(Bp,Au)}, \frac{d(Au,Bp)[d(Tp,Su)+d(Au,Tp)]}{d(Bp,Au)+d(Su,Bp)},\\ \frac{d(Tp,Su)[d(Bp,Au)+d(Su,Bp)]}{d(Tp,Su)+d(Au,Tp)},\\ \frac{d(Tv,Sp)[d(Bp,Tp)+d(Bp,Au)]}{d(Au,Su)+d(Tp,Su)}\end{array}\right\}\right\}^{\lambda}$$

$\leq$

$$\left\{\max\left\{\begin{array}{c}\frac{d(Au,p)[d(Au,Su)+d(p,p)]}{d(Su,p)+d(p,Au)}, \frac{d(Au,p)[d(p,Su)+d(Au,p)]}{d(p,Au)+d(Su,p)},\\ \frac{d(p,Su)[d(p,Au)+d(Su,p)]}{d(p,Su)+d(Au,p)},\\ \frac{d(p,Sp)[d(p,p)+d(p,Au)]}{d(Au,Su)+d(p,Su)}\end{array}\right\}\right\}^{\lambda}$$

$\leq$

$$\left\{\max\left\{\begin{array}{c}\frac{d(Su,p)[d(Su,Su)+d(p,p)]}{d(Su,p)+d(p,Su)}, \frac{d(Su,p)[d(p,Su)+d(Su,p)]}{d(p,Su)+d(Su,p)},\\ \frac{d(p,Su)[d(p,Su)+d(Su,p)]}{d(p,Su)+d(Su,p)},\\ \frac{d(p,Sp)[d(p,p)+d(p,Su)]}{d(Su,Su)+d(p,Su)}\end{array}\right\}\right\}^{\lambda}$$

$=$

$$\left\{\max\left\{\frac{[d^2(p,Sp)+1]}{2}, d(Sp,p), d(Sp,p), d(Sp,p)\right\}\right\}^{\lambda}$$

using (m1) and (m2))

Case 1

If

$\max\left\{\frac{[d^2(p,Sp)+1]}{2}, d(Sp,p), d(Sp,p), d(Sp,p)\right\} = \frac{[d^2(p,Sp)+1]}{2}$

then, equation 2.7 become,

$d(Sp, p) \leq \frac{[d^2(p,Sp)+1]}{2}$, since $\lambda \in (0,1/2)$,

we have , d(Sp, p)=1, which implies that Sp=p.

Case 2

If

$\max\left\{\frac{[d^2(p,Sp)+1]}{2}, d(Sp,p), d(Sp,p), \frac{1}{d(Sp,p)}\right\} = d(Sp,p)$

then, equation 2.7 become,

$d(Sp, p) \leq d^{\lambda}(p, Sp)$, which is a contradiction because, $\lambda \in (0,1/2)$,

and hence we have , d(Sp, p)=1, Sp=p.

This implies that

(2.7) $Sp = Ap = p$.

In similar manner, we consider that $Tp \neq p$ and using the condition (2.2), we get

$$d(p,Tp)=d(Sp,Tp) \leq \left\{\max\left\{\begin{array}{c}\frac{d(Ap,Tp)[d(Ap,Su)+d(Tp,Bp)]}{d(Sp,Tp)+d(Bp,Ap)}, \frac{d(Ap,Bp)[d(Tp,Sp)+d(Ap,Tp)]}{d(Bp,Ap)+d(Sp,Bp)},\\ \frac{d(Tp,Sp)[d(Bp,Ap)+d(Sp,Bp)]}{d(Tp,Sp)+d(Ap,Tp)},\\ \frac{d(Tp,Sp)[d(Bp,Tp)+d(Bp,Ap)]}{d(Ap,Sp)+d(Tp,Sp)}\end{array}\right\}\right\}^{\lambda}$$

$$\leq \left\{\max\left\{\begin{array}{c}\frac{d(p,Tp)[d(p,p)+d(Tp,Tp)]}{d(p,Tp)+d(Tp,p)}, \frac{d(p,Tp)[d(Tp,p)+d(p,Tp)]}{d(Tp,p)+d(p,Tp)},\\ \frac{d(Tp,p)[d(Tp,p)+d(p,Tp)]}{d(Tp,p)+d(p,Tp)},\\ \frac{d(Tp,p)[d(Tp,Tp)+d(Tp,p)]}{d(p,p)+d(Tp,p)}\end{array}\right\}\right\}^{\lambda}$$

(using (2.7))

$$= \{\max\{1, d(p,Tp), d(Tp,p), d(Tp,p)\}\}^{\lambda}$$

$d(p,Tp) \leq d^{\lambda}(p,Tp)$, which is a contradiction because $\lambda \in (0,1/2)$.

Which proves that, Bp = Tp = p.

And hence, Ap=Bp=Sp=Tp=p, i.e p is a common fixed point of A, S, B, T.

This completes the proof.

Uniqueness

Let p and q are two common fixed point such that $p \neq q$, using the equation (2.2)

$$d(Sp,Tq) \leq \left\{\max\left\{\begin{array}{c}\frac{d(Ap,Tq)[d(Ap,Sp)+d(Tq,Bp)]}{d(Sq,Tq)+d(Bq,Ap)}, \frac{d(Ap,Bq)[d(Tq,Sp)+d(Ap,Tq)]}{d(Bq,Ap)+d(Sp,Bq)},\\ \frac{d(Tq,Sp)[d(Bq,Ap)+d(Sp,Bq)]}{d(Tq,Sp)+d(Ap,Tq)},\\ \frac{d(Tq,Sp)[d(Bq,Tq)+d(Bq,Ap)]}{d(Ap,Sp)+d(Tq,Sp)}\end{array}\right\}\right\}^{\lambda}$$

$$\leq \left\{\max\left\{\begin{array}{c}\frac{d(p,q)[d(p,p)+d(q,p)]}{d(q,q)+d(q,p)}, \frac{d(p,q)[d(q,p)+d(p,q)]}{d(q,p)+d(p,q)},\\ \frac{d(q,p)[d(q,p)+d(p,q)]}{d(q,p)+d(p,q)},\\ \frac{d(q,p)[d(q,q)+d(q,p)]}{d(p,p)+d(q,p)}\end{array}\right\}\right\}^{\lambda}$$

$$\leq \left\{\max\left\{\begin{array}{c}d(p,q), d(p,q),\\ d(p,q),\\ d(p,q)\end{array}\right\}\right\}^{\lambda}$$

$d(p,q) \leq d^{\lambda}(p,q)$

which is a contraction. Hence, uniqueness is proved.

References

[1] M. ¨Ozavsar, A.C. C¸ evikel, Fixed points of multiplicative contraction mappings on multiplicative metric spaces, arXiv:1205.5131v1 [math.GM], 2012.
[2] D.R. Sahu, V.B. Dhagat, M. Srivastava, Fixed points with intimate mappings I, Bull. Calcutta Math. Soc., 93 (2001), 107-114.
[3] S.M. Kang, P. Kumar, S. Kumar, P. Nagpa, S.K. Garg, Common fixed points for compatible mappings and its variants in multiplicative metric spaces, Int. J. Pure

Appl. Math., 102 (2015), 383-406. doi: 10.12732/ijpam. v102i2.14718 P. Kumar, S. Kumar, S.M. Kang

[4] A.E. Bashirov, E.M. Kurplnara, A. Ozyapici, Multiplicative calculus and its applications, J. Math. Anal. Appl., 337 (2008), 36-48. doi: 10.1016/j.jmaa.2007.03.081

[5] Parveen Kumar, Sanjay Kumar2, Shin Min Kang3 COMMON FIXED POINTS FOR INTIMATE MAPPINGS IN MULTIPLICATIVE METRIC SPACES, § Volume 103 No. 4 2015, 709-718ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)

[6] M. Sarwar, R. Badshah-e, Some unique fixed point theorems in multiplicative metric space, arXiv:1410.3384v2 [math.GM], 2014

[7] A. Azam, B. Fisher, and M. Khan, "Common fixed point theorems in complex valued metric spaces,"Numerical Functional Analysis and Optimization, vol. 32, no. 3, pp. 243–253, 2011.

[8] Banach ,S.Sur les operations dans les ensembles abstraits et leur application aux equations integrals. Fundam, Math. #, 133-181(1922)

[9] S.M. Kang and Y.P. Kim, Common fixed point theorems, Math. Japonica, 37(6) (1992), 1031-1039 doi:10.1155/2015/218683.

[10] F. Gu, L.M. Cui, Y.H. Wu, Some fixed point theorems for new contractive type mappings, J. Qiqihar Univ., 19 (2013), 85-89.

[11] Florack, L. Assen, HV, multiplicative calculus in biomedical image analysis. J. Math. Imaging vis. 42(1), 64-75 (2012)

[12] M. Abbas, B. Ali, Y. I. Suleiman, Common fixed points of locally contractive mappings in multiplicative metric spaces with application, Int. J. Math. Math. Sci., 2015 (2015), Article ID 218683, 7 pages.