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A NUMERIC INVESTIGATION OF ORDERED AND PARTIALLY-ORDERED VARIANTS OF RAMSEY'S NUMBERS

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Abstract

For a k-uniform hypergraph G with vertex set {1, . . . , n}, the ordered Ramsey number $OR^{k}_{t}(G)$ is the least integer N such that every t-coloring of the edges of the complete k-uniform graph on vertex set {1, . . . , N} contains a monochromatic duplicate of G whose vertices take after the endorsed arrange. Because of this additional request confinement, the requested Ramsey numbers can be considerably bigger than the standard diagram Ramsey numbers. We confirm that the requested Ramsey quantities of freeways under a monotone request develops as a tower of tallness two not as much as the greatest degree as far as the quantity of edges and as a tower of stature one not as much as the most extreme degree as far as the quantity of hues. We additionally broaden hypotheses of Conlon, Fox, Lee, and Sudakov on the requested Ramsey quantities of 2-uniform matchings to give upper limits on the requested Ramsey number of k-uniform matchings under specific orderings.

1. OVERVIEW

Ramsey theory, very generally speaking, is the idea that every structure must contain very well-behaved substructures. In particular, Ramsey theory attempts to find conditions under which specific well-behaved substructures must occur. One of the most basic examples of a problem in Ramsey theory considers coloring the edges of K_6 red and blue. It is a straightforward exercise to observe that any red-blue coloring of the edges of K_6 must contain a monochromatic triangle. To see this, choose any vertex v1, then by the pigeonhole principle [1], there must be at least three other vertices v_2, v_3, v4 that are all connected to v_1 by the same color, say red. If any of the edges between v_2, v_3, v4 are red, say v_2v_3, then $v_1v_2v_3$ forms a red triangle. Otherwise, all of the edges between v_2, v_3, v4 are blue, so $v_2v_3v_4$ forms a blue triangle. Therefore, no matter how the colors are assigned to the edges of K_6, there will always be a monochromatic triangle [2]. Furthermore, Figure 1.1 displays a red-blue coloring of the edges of K_5 that has no monochromatic triangle. Therefore, 6 is the least integer N such that every 2-coloring of the edges of KN contains a monochromatic copy of K_3. The extension of this idea is the crux of Ramsey theory. 1.1 The Graph Ramsey Number Define R_2 (n) to be the least integer N such that every 2-coloring of KN contains a copy of Kn whose edges are all the same color, which is called the 2-color diagonal Ramsey number of n. The argument at the beginning of this chapter shows that R_2 (3) = 6. However, it is not obvious that R_2 (n) is always defined, as it may be possible to color [3]

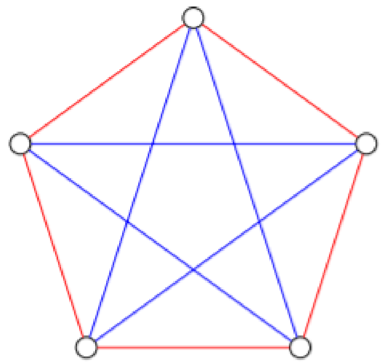

Figure 1: A 2-coloring of K_5 that avoids monochromatic copies of K_3

The edges of K_N in a way that avoids monochromatic copies of K_n for any N. It turns out that R_2 (n) exists for every n due to the following celebrated theorem of the logician Frank P. Ramsey [4].

2. THEOREMS

Theorem 1.1: For any positive integers n and t, there exists another positive integer N such that in any t-coloring of the edges of K_N , there must be a copy of Kn whose edges are all the same color.

With this theorem in mind, we can actually define the t-color diagonal Ramsey number, denoted R_t (n) to be the least integer N such that any t-coloring of $E(K_N)$ admits a copy of K_n whose edges are all the same color. Immediately, we can then define R(n1, . . . , n_t), for positive integers $n_1, \dots, n_t$, called the off-diagonal Ramsey number of n1, . . . , nt , to be the least integer N such that any t-coloring of $E(K_N)$ contains a copy of K_{ni}whose edges all have color i for some i∈ [t]. The existence of $R(n_1, \dots, n_t)$ is seen from the observation that R(n1, . . . , nt) ≤ $R_t(\max\{n_1, \dots, nt\})$ as Km is contained within Kn whenever m ≤ n. Although it was Ramsey who originally developed Ramsey theory, In their monumental 1935 paper [5], along with many other interesting results that we willdiscuss later, Erd˝os and Szekeres proved that

$$R_2(n) \le (1 + o(1)) \frac{4^{n-1}}{\sqrt{\pi n}}.$$

Furthermore, in 1947, proved that

$$R_2(n) \ge (1 - o(1)) \frac{n}{\sqrt{2e}} 2^{n/2}$$

Where e is the base of the natural logarithm. Surprisingly, although the methods used in each of these bounds are not overly complicated, these bounds, roughly $\Omega(2^n/^2) \le R_2(K_n) \le O(2^{2n})$, have remained largely unchanged despite significant effort. Currently, the best bounds on the 2-color diagonal Ramsey number are

$$(1-o(1))\frac{\sqrt{2}n}{e}2^{n/2} \leq \mathrm{R}_2(n) \leq n^{-O(\frac{\log n}{\log\log n})}2^{2n}$$

Due to Spencer [6] and Conlon [7] respectively. Shockingly, Spencer's lower bound is only an improvement over Erd˝os's original bound by a constant factor of 2.

Although most of the focus of Ramsey theory is on the 2-uniform case, there is no reason to restrict ourselves to this case as Ramsey also proved a version of Theorem 1.1 for k-uniform hypergraphs.

Theorem 1.2: For any positive integer's n and t, there exists another positive integer N such that in any t-coloring of the edges of $K\frac{k}{n}$, there must be a copy of $K\frac{k}{n}$ whose edges are all the same color.

Thus, we may extend the definition of the Ramsey number so that $R\frac{k}{t}$(n), the least integer N such that any t-coloring of the edges of $K\frac{k}{n}$ N contains a copy of $K\frac{k}{n}$n whose edges are all the same color, is well-defined. Furthermore, just like the 2-uniform case, we can also consider the off-diagonal case of $R^k(n_1, \ldots, n_t)$. As the 2-uniform case is special, $R^2(n_1, \ldots, n_t) = R(n_1, \ldots, nt)$.

The best bounds on the k-uniform 2-color diagonal Ramsey number are quite loose, especially in comparison to the bounds on the 2-uniform case. The best bounds on $R\frac{k}{n}$ (n) come from a 1965 paper of Erd˝os [8], in which it is shown that

$$\mathrm{tow}_{k-2}(\Omega(n^2)) \leq \mathrm{R}_2^k(n) \leq \mathrm{tow}_{k-1}(O(n)).$$

In fact, it is conjectured that the upper bound is closer to the truth. Interestingly, if it could be shown that $22^{\Omega(n)} \leq R\frac{3}{2}$(n), due to the "stepping up" argument used in [13], then it would automatically hold that$R\frac{k}{2}$ (n) = $\mathrm{tow}_{k-1}(\Theta(n))$ for any k.

In fact, the notion of the Ramsey number naturally extends to any k-uniform hypergraph, as a k-uniform hypergraph on n vertices is a subgraph of $K\frac{k}{n}$.

Formally, a t-coloring of the edges of a k-uniform hypergraph G is a function c : E(G) → [t]. The i-colored subgraph of G is the subgraph of G induced by the edges in c −1 (i). For another hypergraph H, we say that c contains an i-colored copy of H if H is a subgraph of the i-colored subgraph of G.

Definition 1.3. For k-uniform hypergraphs $G_1, \ldots, G_t$, the emphgraph Ramsey number of $G1, \ldots, G_t$, denoted $R^k(G_1, \ldots, G_t)$, is the least integer N such that for any t-coloring of the edges of $K\frac{k}{n}$, there is some i for which there is an i-colored copy of G_i. If $G_1 = \cdots = G_t = G$, then we denote $R^k(G_1, \ldots, G_t)$ by $R\frac{k}{t}$(G) And refer to this as the diagonal case. If not all of the hypergraphs are the same, then we are in the off-diagonal case.

Again, the 2-uniform case is special, so R2 $(G_1, \ldots, G_t) = R(G_1, \ldots, G_t)$.

Notice that we can equivalently define R^k $(G1, \ldots, G_t)$ to be the largest integer N such that there exists a t-coloring of E(K_k N−1) that has no copy of Gi in color i for any i ∈ [t]. Ramsey theory, very basically, asks one question: for hypergraphs G1, . . . ,Gt , what is R k (G1, . . . , Gt)? In order to answer this question, we must call upon both definitions of the Ramsey number.

If we want to show that Rk (G_1,G_t) ≤ N1, then we must show that any t-coloring of E ($K_k N_1$) contains an i-colored copy of G_i for some i. On this other hand, if we wish to show that Rk (G1, . . . , G_t) ≥ N^2, we must demonstrate a t-coloring if E(K_k N^2−1) that avoids i-colored copies of Gi for each i. Arrow Notation The original formulation of Ramsey theory is tied to coloring the complete graph and looking for monochromatic substructures. Because any vertex in the complete graph is indistinguishable from any other vertex, "na¨ıve" techniques such as the pigeonhole principle can easily be applied to achieve bounds on the Ramsey numbers. However, it is natural to ask if we can define some analogue of Ramsey theory which instead considers coloring graphs other than the complete graph.

The answer, of course, is yes and is done by defining what is known as arrow notation. For k-uniform hypergraphs H, G^1, . . . , G_t , we say that H k −→ ($G^{1,}$ . . . , G_t) if any tcoloring of E(H) admits an i-colored copy of Gi for some i ∈ [t]. We refer to H as the host graph. Using this notation, we can define the Ramsey number as follows: R k (G^1,G^t) = min {|V (H)|: H k −→ (G1,Gt)}.

This is equivalent to the previous definition of the Ramsey number because if any tcoloring of E(H) contains an i-colored copy of G_i for some i, then so does any t-coloring of E(Kk |V (H)|). Arrow notation can be used to define Ramsey-type numbers for many different parameters of the host graph other than just the number of vertices. An interesting Ramseytype number which is defined through arrow notation is called the size Ramsey number of G_1, . . . , G_t , which is defined to be min{|E(H)| : H k −→ (G^1, . . . , G_t)}. It is easy to observe that the size Ramsey number is bounded above by Rk (G1,...,G_t) k as this is the number of edges in the complete k-uniform graph of order Rk (G_1, . . . , G^t); however, in many cases, the size Ramsey number can be much smaller. Using arrow notation, we can also define Ramsey-type numbers for different host families of graphs. For example, we could look at the family of hypercube graphs, Q_n,

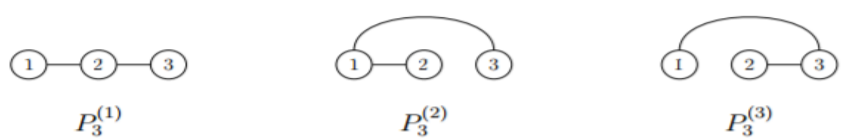

Figure 1 Three no isomorphiclabeling of P_3

And given graphs G_1, . . . , G_t that can appear as subgraphs of a hypercube, we can try to determine the least integer N such that QN 2 −→ (G_1, . . . , G_t). While we will not discuss the notion of size Ramsey numbers or Ramsey numbers on the hypercubes, we will return to arrow notation in Chapter 3 in order to define partially-ordered Ramsey numbers in full generality [9]

3. THE DIRECTED RAMSEY NUMBER

As Ramsey theory grew in popularity among discrete mathematicians, it was quickly

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realized that even seemingly simple questions were very challenging. Because of this, variants of Ramsey numbers were introduced both as possible stepping stones to these problems and as independently interesting concepts [10].

In this paper, we focus on recent variants of Ramsey theory that consider graphs whose vertex sets are ordered in some fashion.

Consider the 2-uniform path on 3 vertices, P_3. By the pigeonhole principle, it is immediate to note that $R_t(P_3) \leq t + 2$ as if c is a t-coloring of $E(K_N)$ that avoids monochromatic copies of P_3[11], then no vertex can be incident to two edges of the same color.

Now consider labeling on the vertices of P3 with the set {1, 2, 3} (see Figure 1). We can now ask the following question:

Fix an ordering of the vertices of K_N (i.e. consider K_N to have vertex set [N]), and color the edges; how large can N be so that I avoid monochromatic copies of a particular

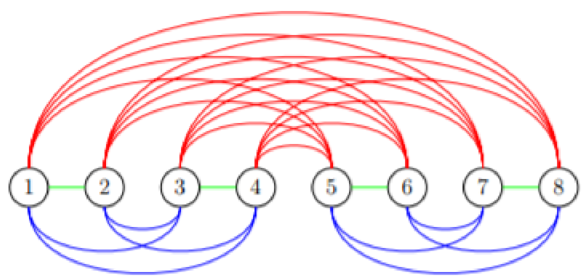

Figure 2 A 3-coloring of E (K8) that avoids monochromatic copies of P (1)

Ordering of a given graph G? For example, Figure 2 displays a 3-coloring of E(K8) that avoids monochromatic copies of $P^{(1)}{}_3$; however, this coloring admits many monochromatic copies of $P_3{}^{(2)}$ and $P_3{}^{(3)}$. In fact, the pattern shown in Figure 2 can be repeated to show that there is a t-coloring of $E(K_2{}^t)$ that does not have any monochromatic copies of $P_3{}^{(1)}$. On the other hand, any t-coloring of $E(K_{t+2})$ must admit monochromatic copies of both $P_3{}^{(2)}$ and $P_3{}^{(3)}$. Thus, we may be tempted to say that $Rt(P_3{}^{(2)}) = Rt(P_3{}^{(3)}) \leq t + 2$ while $Rt (P^{(1)} 3) > 2 t$.

In 2002 [12], introduced the first formalization of this idea through the concept of the directed Ramsey number[1]

The transitive tournament of order n, denoted TT_n, is a directed-acyclic orientation of K_n. In other words, TT_n has the property that for every $\{x_1, x_2\} \in V (TT_n)$, either $(x_1, x_2) \in E(TT_n)$ or $(x_2, x_1) \in E(TT_n)$, and if $(x_1, x_2),(x_2, x_3) \in E(TT_n)$, then $(x_1, x_3) \in E(TT_n)$.

For directed-acyclic digraphs D1, . . . , Dt , the directed Ramsey number of $D_1, . . . , D_t$, denoted $DR(D_1, . . . , D_t)$, is the least integer N such that any t-coloring of E(TTN) contains a copy of D_i in color i for some i. The fact that this number exists follows from the simple observation that$DR_t (TT_n) = R_t(n)$

4. THE ORDERED RAMSEY NUMBERS

An alternative formalization, called ordered Ramsey theory, has recently received significant attention [13]. In this variation, we again look for t-colorings of the complete graph that avoid monochromatic copies of a graph G, except that the order of the vertices of G in this monochromatic copy is very important.

Formally, ordered k-uniform hypergraph is a hypergraph G where the edge set E(G) contains k-sets of vertices, and the vertex set V (G) is totally ordered. An ordered hypergraph G is contained in an ordered hypergraph H if there is an injective, order preserving map from the vertices of G to the vertices of H such that edges of G map to edges of H. Let Kk N be the complete k-uniform hypergraph on the vertex set {1, . . . , N} and let c : E($K\frac{k}{n}$) → {1, . . . , t} be a t-coloring of the edges in $K\frac{k}{N}$. The i-colored subgraph of $K\frac{k}{n}$ is the ordered hypergraph given by the edges in c −1 (i).

For ordered k-uniform hypergraphs G1, . . . ,Gt , the ordered Ramsey number

OR^k (G1, . . . ,Gt) is the minimum N such that for every t-coloring of $K\frac{k}{n}$ there is some color i such that the i-colored subgraph contains G_i . This number is necessarily defined and finite, since there exists an n such that each Gi is a subgraph of $K\frac{k}{n}$ and hence ORk (G1, . . . , Gt) ≤ R $\frac{k}{t}$(n). If G1 = · · · = Gt = G, then we denote OR^k (G1, . . . ,Gt) as $OR\frac{k}{t}$(G) and refer to this as the diagonal case; otherwise it is the off-diagonal case.

Notice that each ordered graph gives rise to a directed graph in a natural way. If G is an ordered graph, form the digraph Gby letting (x, y) ∈E(G0) whenever {x, y} ∈ E(G) and x < y. Thus, it is easy to observe that ORt (G) ≥ DRt (G0). However, the opposite inequality need not hold. This follows from the fact that for a given digraph G, there

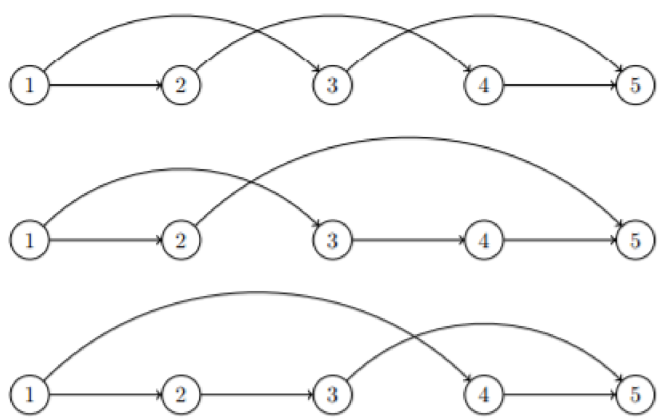

Figure 3Nonisomorphic orderings of the same digraph.

May be multiple nonisomorphic orderings of the vertex set such that x < y whenever (x, y) ∈E(G) (see Figure 3). Because of this, the ordered Ramsey number should be viewed as the “proper” way to extend the notion of Ramsey numbers to graphs with an order on their vertices. If G is a 2-uniform path under the standard ordering, then the 2-color ordered Ramsey number of G is equal to the bound of the Erd˝os-SzekeresTheorem [14], and if G is a tight 3-uniform path under the standard ordering, then the 2-color ordered Ramsey number of G is equal to the bound of the happy ending problem [15]. Due to these connections, much of the previous work has focused on the ordered Ramsey number of tight k-uniform paths under the standard ordering [16].

Applications of the Ordered Ramsey Number

Although the formal definition of ordered Ramsey numbers is fairly new, the idea has been around since the monumental 1935 [17]. We briefly present the connections between Erd˝os-Szekeres type problems and the ordered Ramsey numbers of hyperpaths.

For positive integers k,l\`, e such that k >L, the (k,-l\`)-path on e edges, denoted $P^{k}{}_{,e}$, is the k-uniform ordered hypergraphon e(k − \`l) + \` vertices and e totally-ordered edges $A_1, A_2, \ldots, A_e$ where two consecutive edges A_i , A_i+_1 intersect exactly on the maximum \` vertices in Ai and the minimum \` vertices in Ai+1. The path $P_e{}^k{}_{,}{}^k{-L}$ e is called the tight k-uniform path and otherwise $P^{k\text{-}}$L,\` e is a loose path.

Erd˝os-Szekeres Type Problems

In 1935, Erd˝os and Szekeres [14] proved that any sequence of (n − 1)2 + 1 distinct real numbers must contain either an increasing or a decreasing subsequence of length n. The original proof of this fact had an inductive flavor, and there have since appeared very slick proofs that require only an elementary application of the pigeonhole principle. In addition to these, there is a very natural connection of this problem to the ordered Ramsey numbers of paths

5. CONCLUSION

Our investigation into arbitrarily-ordered k-uniform matching's provides upper bounds that are similar to the previous bounds in the 2-uniform case. Extending the techniques from 2-uniform matching's comes at the cost that it does not apply to all k-uniform ordered matching's, but they do provide bounds that are exponential and not a tower. However, our methods do not allude to lower bounds, and hence it is unclear whether our upper bounds are tight

The largest question left open from our study of ordered Ramsey numbers is related to arbitrary orderings of (k,l) –paths While we found upper bounds on $OR_t(P_e{}^{2,1}$), for arbitrary orderings of P $k_e{}^k$,l would be very interesting and would significantly extend our current techniques.

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