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TANGENTIAL CAUCHY–RIEMANN EQUATIONS ON GEOMETRY SUBMANIFOLDS AND ITS APPLICATIONS: A STUDY

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Abstract

This research focused on contributions by several experts in the theory of isometric immersions between Riemannian manifolds and focuses on the geometry of CR structures on submanifolds in Hermitian manifolds. CR structures are a bundle theoretic recast of the tangential Cauchy–Riemann equations in complex analysis in several complex variable are available to enhance the geometry submanifolds in hermitian manifold geometry. A uniqueness part is in like manner illustrated. An undifferentiated from result, without the uniqueness part, is in like manner exhibited for C°° submanifolds. We have dealt with the examination as seeks after: In §2 the geometric thoughts, for instance, the Levi structure and the Levi variable-based math are explained from a Pfaffian framework (differential structures) point of view

1. OVERVIEW

The theory of complex manifolds dates back many decades so that its origins are considered classical even by the standards of mathematicians. Consequently, there are many fine references on this subject. By contrast, the origins of the theory of CR manifolds are much more recent even though this class of manifolds contains very natural objects of mathematical study (for example, real hypersurfaces in complex Euclidean space). The first formal definition of the tangential Cauchy-Riemann complex did not appear until the mid-1960s with the work. of Kohn and Rossi. From that point forward, CR manifolds and the tangential Cauchy-Riemann complex have been broadly contemplated both for their characteristic intrigue and in view of their application to different fields of concentrate, for example, halfway differential equations and numerical material science. The motivation behind this book is to characterize CR manifolds and the related tangential Cauchy-Riemann complex and to talk about a portion of their fundamental properties. What's more, we will test a portion of the significant late advancements in the field (up to the mid-1980s) [1].

Over the most recent two decades, inquire about about CR manifolds has extended into numerous territories. Two of these regions that are important to us are (l) the holomorphic augmentation of CR functions (answers for the homogeneous tangential Cauchy-Riemann equations) and (2) the nearby feasibility or non-reasonability of the tangential Cauchy-Riemann complex. The first area started in the 1950s when Hans Lewy showed that under certain

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convexity assumptions on a real hypersurface in en, CR functions locally extend to holomorphic functions. Over the years, many refinements have been made to this CR extension theorem so that it now includes manifolds of higher codimension with weaker convexity assumptions. The second area started in the 1960s with the work of Kohn. He utilized a Hilbert space (.C2) way to deal with develop answers for the tangential Cauchy-Riemann complex on the limit of a carefully pseudo arched area (aside from at top degree). Afterward, Henkin created indispensable pieces to speak to answers for the tangential Cauchy-Riemann equations [2-4].

A firmly related point is the non-resolvability of specific frameworks of halfway differential equations. During the 1950s, Hans Lewy built a case of a halfway differential equation with smooth coefficients that has no privately characterized smooth arrangement. Specifically, he demonstrated that "C00" can't supplant "real analytic" in the announcement of the Cauchy-Kowalevsky theorem. Lewy's model is firmly identified with the tangential Cauchy-Riemann equations on the Heisenberg group in C2. His model outlines that the tangential Cauchy-Riemann complex on a carefully pseudo raised limit isn't constantly reasonable at the top degree. Afterward, Henkin built up a basis for feasibility of the tangential Cauchy-Riemann complex at the top degree.

The present volume assembles commitments by a few specialists in the theory of isometric submersions between Riemannian manifolds and spotlights on the geometry of CR structures on submanifolds in Hermitian manifolds. CR structures are a bundle theoretic recast of the tangential Cauchy–Riemann equations in complex analysis in a few complex factors. Let X Cn (n ≥ 2) be an open set and let.

$$\bar{\partial} f \equiv \frac{\partial f}{\partial \bar{z}^j} d\bar{z}^j = 0 \qquad (1)$$

be the ordinary "Cauchy–Riemann equations in Cn . A function $f \in C^1(\Omega, \mathbb{C})$ is holomorphic in Ω if f satisfies (1) everywhere in Ω. Let M be an embedded real hypersurface in Cn such that $U = M \cap \Omega \neq \emptyset$ " and let us set

$$T_{1,0}(M)_x = [T_x(M) \otimes_{\mathbb{R}} \mathbb{C}] \cap T^{1,0}(\mathbb{C}^n)_x, \quad x \in M, \qquad (2)$$

where "T1;0ðCn Þ is the holomorphic tangent bundle over Cn (the span of $\{\partial/\partial z^j : 1 \leq j \leq n\}$). Then $T_{1,0}(M)$ is a rank n − 1 complex vector bundle over M, referred to as the CR structure of M (induced on M by the complex structure of the ambient space Cn) and one may consider the first order differential operator"

$$\overline{\partial}_b : C^1(M,\mathbb{C}) \to C\left(T_{0,1}(M)^*\right), \tag{3}$$

$$(\overline{\partial}_b u)\overline{Z} = \overline{Z}(u), \quad u \in C^1(M,\mathbb{C}), \quad Z \in T_{1,0}(M),$$

"(the tangential Cauchy–Riemann operator) where $T_{0,1}(M) = \overline{T_{1,0}(M)}$ (overbars denote complex conjugates). A function $\in C^1(M,\mathbb{C})$ is a CR function on M if u satisfies"

$$\overline{\partial}_b u = 0 \tag{4}$$

"(the tangential Cauchy–Riemann equations) everywhere on M. Let $\mathrm{CR}^1(M)$ be the space of all CR functions on M. The trace on U of any holomorphic function $f \in \mathcal{O}(\Omega)$ is a CR function $u \in \mathrm{CR}^1(U)$. In other words, the Cauchy–Riemann equations (1) induce on M the first order partial differential system (4). A sufficiently small open piece U of M may be described by a smooth defining function $\rho : \Omega \to \mathbb{R}$ "i.e.,

$$U = \{x \in \Omega : \rho(x) = 0\}$$

such that " $D\rho(x) \neq 0$ for any $x \in U$. By eventually restricting the open set U we may assume that $\rho_{z^n}(x) \neq 0$ for any $x \in U$. Here $\rho_{z^j} \equiv \partial\rho/\partial z^j$ for $1 \leq j \leq n$. The portion of $T_{1,0}(M)$ over U" is then the span of

$$Z_\alpha \equiv \rho_{z^n}\frac{\partial}{\partial z^\alpha} - \rho_{z^\alpha}\frac{\partial}{\partial z^n}, \quad 1 \leq \alpha \leq n-1,$$

and the tangential Cauchy–Riemann equations (4) on U may be written as

$$Z_{\overline{\alpha}}(u) = 0, \quad 1 \leq \alpha \leq n-1, \tag{5}$$

where $Z_{\overline{\alpha}} \equiv \overline{Z_\alpha}$. Accordingly the tangential Cauchy–Riemann equations might be believed to be a first order overdetermined PDEs framework with smooth complex esteemed coefficients. While consistent coefficient equations are these days genuinely surely known, there is still much work to do on factor coefficient PDEs, for example, (5).

"The geometric approach to the study of (local and global properties of) solutions to (4) or (5) is to study the complex vector bundle $T_{1,0}(M)$. This is commonly accomplished by introducing additional geometric objects, familiar within differential geometry. For instance, should one need

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to compute the Chern classes of $T_{1,0}(M)$, one would need a connection in $T_{1,0}(M)$. Indeed it is rather well known (cf. e.g., [132]) that Tanaka and Webster built (cf. [133, 134]) a linear connection ∇ on any nondegenerate real hypersurface $M \subset \mathbb{C}^n$, uniquely determined by a fixed contact from Θ on M [the Tanaka–Webster connection of (M, Θ)] and such that r descends to a connection in $T_{1,0}(M)$ as a vector bundle. Chern classes of $T_{1,0}(M)$ may then be computed in terms of the curvature of the Tanaka–Webster connection, in the presence of a fixed contact form on M".

"CR structures induced on real hypersurfaces of C^n are but a particular instance of a more general notion, that of an abstract CR structure on a $(2n+k)$ -dimensional manifold M. A complex subbundle $T_{1,0}(M) \subset T(M) \otimes \mathbb{C}$, of complex rank n, of the complexified tangent bundle, is said to be an (abstract) CR structure on M" if

$$T_{1,0}(M)_x \cap T_{0,1}(M)_x = (0), \quad x \in M,$$

$$Z, W \in C^\infty(U, T_{1,0}(M)) \Rightarrow [Z, W] \in C^\infty(U, T_{1,0}(M)), \qquad (7)$$

for any open subset $U \subset M$. "The integers n and k are respectively the CR dimension and CR codimension of $T_{1,0}(M)$ while the pair ðn; kÞ is its type. A complex sub-bundle $T_{1,0}(M) \subset T(M) \otimes \mathbb{C}$ fulfilling just aphorism (6) is a near CR structure on M. Saying (7) is frequently alluded to as the formal (or Frobenius) integrability property. A nearly CR structure fulfilling the formal integrability property (7) is a CR structure. CR structures on real hypersurfaces $M \subset \mathbb{C}^n$ have type (n – 1, 1)".

An enormous part of the numerical writing dedicated to the investigation of CR structures is restricted to the case of CR codimension 1 within the sight of extra nondegeneracy suspicions (cf. [5]). Let $(M, T_{1,0}(M))$ be a CR manifold of type (n, k). The Levi distribution is the real rank 2n distribution

$$H(M) = \mathrm{Re}\{T_{1,0}(M) \oplus T_{0,1}(M)\}.$$

It carries the complex structure

$$J : H(M) \to H(M), \quad J(Z + \bar{Z}) = i(Z - \bar{Z}), \; Z \in T_{1,0}(M),$$

$(i = \sqrt{-1})$. The Levi form of $(M, T_{1,0}(M))$ is

$$L_x : T_{1,0}(M)_x \times T_{0,1}(M)_x \to \frac{T_x(M) \otimes_{\mathbb{R}} \mathbb{C}}{H(M)_x \otimes_{\mathbb{R}} \mathbb{C}}, \quad x \in M,$$

$$L_x(z,w) = i\pi_x\left[Z, \overline{W}\right]_x, \quad z, w \in T_{1,0}(M)_x,$$

"where $Z, W \in C^\infty(T_{1,0}(M))$ are arbitrary globally defined smooth sections such that $Z_x = z$ and $W_x = w$. Also $\pi : T(M) \otimes \mathbb{C} \to [T(M) \otimes \mathbb{C}]/[H(M) \otimes \mathbb{C}]$ is the natural projection. The CR structure $T_{1,0}(M)$ [or the CR manifold $(M, T_{1,0}(M))$] is nondegenerate if Lx is nondegenerate for any $x \in M$. Assuming that k = 1 there is yet another customary description of the Levi form and of nondegeneracy, as understood in complex analysis".

Let

$$H(M)_x^\perp = \{\omega \in T_x^*(M) : \mathrm{Ker}(\omega) \supset H(M)_x\}, \quad x \in M,$$

be the conormal bundle associated to HðMÞ. "Assume that M is oriented, so that T(M)is oriented as a vector bundle. The Levi distribution H(M) is oriented by its complex structure J. Hence the quotient $T(M)/H(M)$ is oriented. There are (noncanonical) bundle isomorphisms $H(M)^\perp \approx T(M)/H(M)$, hence $H(M)^{-1}$ is oriented, as well. Any oriented real line bundle over a connected manifold is trivial, hence $H(M)^\perp \approx M \times \mathbb{R}$ (a vector bundle isomorphism). Hence globally defined nowhere zero smooth sections $\theta \in C^\infty(H(M)^\perp)$ [referred to as pseudohermitian structures on M]" do exist.

Let P be the set of all pseudohermitian structures on M. Given $\theta \in \mathcal{P}$ one may set

$$L_\theta(Z, \overline{W}) = -i(d\theta)(Z, \overline{W}), \quad Z, W \in T_{1,0}(M).$$

and one may easily check that L_Θ and L agree. As it turns out, if $T_{1,0}(M)$ is nondegenerate then each $\theta \in \mathcal{P}$ is a contact form, i.e., $\theta \wedge (d\theta)^n$ is a volume form on M.

Let $(M, T_{1,0}(M))$ be a nondegenerate CR manifold, of type (n, 1), and let $\theta \in \mathcal{P}$. The Reeb vector of (M, θ) is the unique globally defined nowhere zero tangent vector field n $\xi \in \mathfrak{X}(M)$ determined by

$$\theta(\xi) = 1, \quad \xi \rfloor d\theta = 0.$$

"The Webster metric is the semi-Riemannian metric g_θ on M given by

$$g_\theta(X,Y) = (d\theta)(X,JY), \quad g_\theta(X,\xi) = 0, \; g_\theta(\xi,\xi) = 1, \qquad (8)$$

for any $X, Y \in H(M)$. Axioms (8) uniquely determine g_θ because of $T(M) = H(M) \oplus \mathbb{R}\xi$." Rn. For any nondegenerate CR manifold $(M, T_{1,0}(M))$, on which a contact form h 2 P has been specified, there is a unique linear connection ∇ on M [the Tanaka–Webster connection of (M,θ) obeying to the following axioms (i) H(M) is parallel with respect to ∇, (ii) $\nabla J = 0$ and $\nabla g_\theta = 0$, and (iii) the torsion tensor field T_∇ is pure, i.e.,

$$T_\nabla(Z,W) = 0, \quad T_\nabla(Z,\overline{W}) = 2iL_\theta(Z,\overline{W})\xi,$$

$$\tau \circ J + J \circ \tau = 0,$$

for any $Z, W \in T_{1,0}(M)$, where $\tau(X) = T_\nabla(\xi, X)$ for any $X \in \mathfrak{X}(M)$.. One should notice that the existence of r is tied to that of n, which in turn is a direct consequence of nondegeneracy and orientability.

We say "$(M, T_{1,0}(M))$ is strictly pseudo convex if L_θ is positive definite for some $\theta \in \mathcal{P}$. To emphasize on the role play by Chern classes $c_j(T_{1,0}(M))$, let us recall (cf. [6, 7]) the Lee conjecture according to which any abstract strictly pseudo convex CR manifold M with $c_1(T_{1,0}(M)) = 0$ should admit a contact form h such that (M,θ) is pseudo-Einstein [i.e., the pseudohermitian Ricci tensor (of the Tanaka– Webster connection of (M,θ)) is proportional to the Levi form]. The sphere $S^{2n-1} \subset \mathbb{C}^n$ is pseudo-Einstein with the canonical contact form $\theta = \frac{i}{2}\left(\overline{\partial} - \partial\right)|z|^2$ [and of course $c_1\left(T_{1,0}(S^{2n-1})\right) = 0$]".

"From the definition of the notion of an (abstract) CR structure, it is manifest that the prospective study of the CR structure $T_{1,0}(M)$ of a real hypersurface $M \subset \mathbb{C}^n$ ignores the metric structure (the canonical Euclidean structure of $\mathbb{C}^n \approx \mathbb{R}^{2n}$). and only takes into consideration the complex structure on C^n. Whatever metric structure M is seen to possess a posteriori, such as the Levi form L_θ, springs from the CR structure (from the complex structure J along $H(M)$) and is determined by it only up to a "conformal factor", very much as in the theory of Riemann surfaces"..

Indeed "if $\theta, \hat{\theta} \in \mathcal{P}$ then $\hat{\theta} = \lambda\theta$ for some C^∞ function k : M ! Rnf0g, implying that $\lambda : M \to \mathbb{R}\backslash\{0\}$. However, the fact that the Webster metric g_θ is semi-Riemannian is tied to nondegeneracy (g_θ is actually Riemannian when $(M, T_{1,0}(M))$ is strictly pseudoconvex) and none of these objects, including of course the Tanaka–Webster connection, is available on a CR manifold whose Levi form has a degeneracy locus (for instance in the extreme case where $(M, T_{1,0}(M))$ is Levi flat, i.e., $L_\theta = 0$ for some $\theta \in \mathcal{P}$, and thus for all)". We likewise underline that everything said and done in pseudohermitian geometry is limited to the beginning presumption that the given CR manifold has CR codimension $k = 1$.

2. "LEVI GEOMETRY AND THE TANGENTIAL CAUCHY-RIEMANN EQUATIONS ON A REAL ANALYTIC SUBMANIFOLD OF C"

The connection between the Levi geometry of a submanifold of C" and the tangential Cauchy-Riemann equations is contemplated. On a real analytic codimension two submanifold of C", we discover conditions on the Levi variable-based math which enable us to locally understand the tangential Cauchy-Riemann equations (in many bidegrees) with parts. Under similar conditions, we demonstrate that, locally, any CR-function is the limit worth hop of a holomorphic function characterized on some appropriate open set in C". This limit worth hop result is the most ideal outcome since we likewise demonstrate that there is nobody sided augmentation theory for such submanifolds of C". Truth be told, we demonstrate that if 5 is a real analytic, conventional, submanifold of C" (any codimension) where the overabundance dimension of the Levi variable based math is not exactly the real codimension, at that point S isn't extendible to any open set in C"

3. CONCLUSION

Starting late, there has been growing excitement for the relationship of geometric thoughts, for instance, the Levi kind of a hypersurface and the area resolvability of the tangential Cauchy-Riemann equations.

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