Inequalities For The Polar Derivative of a Polynomial

RAJKALA

Asst. Prof. Dr. B.R.A.G.C.Kaithal-136027

Abstract :- The geometry of polynomials explores geometrical relationships between the zeros and the coefficients of a polynomial. A classical problem in this theory is to locate the zeros of a given polynomial by determining disks in the complex plane in which all its zeros are situated. In this paper, we infer bounds for general polynomials and apply classical and new results to graph polynomials namely Wiener and distance polynomials whose zeros have not been yet investigated. Also, we examine the quality of such bounds by considering four graphs classes and interpret the results.

Keywords:- Polynomial explores, Complex plane investigated, chomatic polynomial

Introduction

Numerous graph polynomials have been extensively studied and applied inter disciplinarily, see, e.g.,[1]–[4]. Early contributions in this area deal with studying the well known independence polynomial [5]and chromatic polynomial [6]. Other graph polynomials such as the Omega polynomial and Cluj polynomial have been studied in [7]. Apart from this research, polynomials have been also employed in biologically driven disciplines. For instance, Emmert-Streib [8] tackled the challenging problem of calculating knot polynomials of secondary structure elements of proteins algorithmically. Related work can be also found in [8]. Interestingly, the development of so-called topological indices such as the well-known Wiener index [9] has triggered exploring graph polynomials too. For instance, Yan et al. [10]examined how the Wiener index changes under certain graph operations and extended their results to Wiener polynomials. Zadeh et al. [11] also investigated Wiener-type invariants of some graph operations. But note that the first paper exploring the change of the Wiener number upon operations on graphs has been contributed by Polansky and Bonchev [12]. Further, formulas for the Wiener polynomial of k-th power graphs have been investigated [13] when considering special graph classes such as paths, cycles and hypercubes (see also Theorem (1)).

In general, graph polynomials have been developed for measuring combinatorial graph invariants and for characterizing graphs. The latter problem has been studied in structural chemistry where the polynomials have been derived from chemical graphs [1], [3]. There, graphs have been characterized by several graph polynomials [14] to solve problems in the Hückel-molecular orbital theory and in the theory of aromaticity, see [3], [15]. Another intriguing field deals with investigating graph measures derived from the zeros of a graph polynomial. Seminal work has been done by Lovász et al. [16] as they explored the meaning of the largest eigenvalue of trees. Particularly they found that the leading positive eigenvalue of the characteristic polynomial can be used as a measure for detecting branching of trees. Related concepts of branching based on using the eigenvalues of a graph have been studied by Randić et al. [17] and Bonchev [18].

Later, Randić et al. [17] surveyed further eigenvalue-based measures such as the sum of the positive eigenvalues, the multiplicity of the zero eigenvalue and other spectral indices [17], [19]. Also, Dehmer et al. [20] recently developed novel spectral measures that turned to be unique for several graph classes. Altogether this shows that graph polynomials and their zeros have been a valuable source for investigating various problems in discrete mathematics and related areas.

Apart from the research described above, the zeros of some graph polynomials have been also explored, see, e.g., [21]–[23]. In this sense, Woodall [23] explored the zeros and zero-free regions of chromatic and flow polynomials. Also, the zero distribution of chromatic and flow polynomials of graphs and characteristic polynomials of matroids have been examined by Jackson [21]. Finally Brešar et al. [24]examined the zeros of cube polynomials under certain structural conditions of the underlying graphs. Other results about the zeros of known graph polynomials have been recently reported by Ellis-Monaghan et al. [4].

The main contribution of this paper is twofold: First, we prove inclusion radii representing upper bounds for the zeros of general complex polynomials. Note that most of these statements can also be applied if the polynomials possess real coefficients as the moduli of the coefficients appear in the corresponding bounds. Second, we apply these and classical results to locate the zeros of special Wiener and distance polynomials, see [25]–[27]. This results in disks in the complex plane or intervals where the zeros of these polynomials lie. To our best knowledge, the location of zeros of the Wiener and distance polynomial has not been studied yet. Apart from proving results for special polynomials, i.e., the polynomials represent special graph classes, it is easy to generalize the results for other (general) graph polynomials by using the tools we will provide in this paper. Besides further developing the mathematical apparatus, we evaluate the quality of the zero bounds by generating four large graph classes and interpret the numerical results.

Results

The main contribution of this paper is to locate the zeros of special graph polynomials which have been proven useful in mathematical chemistry and discrete mathematics, see [3], [14], [25]. A thorough overview of the underlying theory called *analytic theory of polynomials* can be found in [28], [29]. Note that the problem of finding bounds for the zeros of complex and real polynomials has been tackled by numerous authors, e.g., see [28], [30]–[34]. However, the existing research shows that the usefulness and performance of many such bounds has not been demonstrated yet. For this, we compare our bounds in the section 'Numerical Results' and demonstrate that some of the new bounds are optimal.

We now start by reproducing some important definitions and results we are going to use in our analysis.

Mathematical Preliminaries

In this section, we introduce some mathematical preliminaries [25]– [27], [35], [36]. Let $G=(V, E)$ be a finite simple graph and let $A=(a_{ij})$ be its adjacency matrix. I denotes the identity matrix. Then,

$$P_{G,A}(z) := \det(A-zI),$$

(1)

is the characteristic polynomial of G. Straightforwardly, we obtain the distance polynomial defined by

$$P_{G,D}(z) := \det(D-zI),$$

(2)

where D is the distance matrix of G. By expanding the determinant, we yield

(3)

$$P_{G,D}(z)=z^n+a_{n-2}z^{n-2}+\cdots+a_1z+a_0.$$

We see that s_{n-1} is always equal to zero [26]. Denote by $\rho(G)$ the diameter of G and $d(G,i)$ is the number of pairs of G having distance i, $d(G,1)=|E|$. Then the Wiener polynomial [25], [27] (also called Hosoya polynomial [37]) can be defined as

$$W_G(z) := \sum_{i=1}^{\rho(G)} d(G,\ i)z^i.$$

(4)

Further properties of $W_G(z)$ have been reported in [27]. Next, we reproduce some results due to Sagan et al. [27] and K ivka [26] giving concrete expressions for Wiener- and distance polynomials for special graph classes.

Theorem 1 *Let* P_n, C_n *and* Q_n *be the path graph, cycle graph and* n*-dimensional cube. It holds*

$$W_{P_n}(z) = (n-1)z + (n-2)z^2 + \cdots + z^{n-1},$$

(5)

$$W_{C_{2n}}(z) = (2n)(z + z^2 + \cdots + z^{n-1}) + nz^n,$$

(6)

$$W_{C_{2n+1}}(z) = (2n+1)(z+z^2+\cdots+z^n),$$

(7)

$$W_{Q_n}(z) = 2^{n-1}\{(1+z)^n - 1\}.$$

(8)

Theorem 2 *Let* K_n *and* S_n *be the complete graph and the star graph on vertices. It holds*

$$P_{K_n,D}^{\;n}(z) = (z+1)^n(z-n+1),$$

(9)

$$P_{S_n,D}(z) = (z+2)^{n-2}[z^2-1-(n-2)(2z+1)].$$

(10)

To introduce the problem of locating the zeros of polynomials, we state the following definitions.

Definition 1 *Let*

$$f(z) = \sum_{i=0}^{n} a_i z^i, a_n \neq 0, a_i \in \mathbb{C}, i = 0, 1, \ldots, n,$$

(11)

be complex polynomial. The set

$$K(z_0,\, r) := \{z \in \mathbb{C} \,|\, |z - z_0| \leq r\},$$

(12)

represents a circle with central point z_0 *and radius* r*. Further, we define*

$$\hat{K}(z_0,\, r) := \{z \in \mathbb{C} \,|\, |z - z_0| < r\}.$$

(13)

Definition 2 *If all zeros of* $f(z)$ *lie in the set given by Equation (12),* r *is called the inclusion radius. In the simplest case,* r *is a function of all coefficients, i.e.,* $r = r(a_0, a_1, \ldots, a_n)$*.*

Note that a more general question namely deriving bounds depending on $p+1$ coefficients for p zeros of $f(z)$ has been tackled by Montel [28], [38]. Other variants of bounds and extensions of the results due to Montel can be also found in [28].

Known Inclusion Radii

In this section, we state some classical and known results for locating the zeros of arbitrary complex-valued polynomials.

Theorem 3 (Cauchy [28]) *Let*

$$f(z)=\sum_{i=0}^{n} a_i z^i, a_n \neq 0, i=0,1,\ldots,n,$$

(14)

be complex polynomial. All zeros of $f(z)$ *lie in* $K(0,1+M_1)$*, where*

$$M_1 := \max_{0\leq j\leq n-1}\left|\frac{a_j}{a_n}\right|.$$

(15)

Theorem 4 (Fujiwara [39]) *Let*

$$f(z)=\sum_{i=0}^{n} a_i z^i, a_n \neq 0, i=0,1,\ldots,n,$$

(16)

be complex polynomial. For $\lambda_1,\cdots,\lambda_n>0$ *and* $\sum_{j=1}^{n}\frac{1}{\lambda_j}\leq 1$*, all zeros of* $f(z)$ *liein*

$$K\left(0,1+\max_{1\le j\le n}\left(\frac{|a_{n-j}|\lambda_j}{|a_n|}\right)^{\frac{1}{j}}\right).$$

(17)

Theorem 5 (Enestrom-Kakeya [40]) *Let*

$$f(z)=a_nz^n+a_{n-1}z^{n-1}+\cdots+a_0, a_i\in \mathrm{IR}, i=0,1,\ldots,n,$$

(18)

be a polynomial with real coefficients satisfying

$$a_0\ge a_1\ge\cdots\ge a_n>0.$$

(19)

Then, no zeros of f(z) lie in $\hat{K}(0,1)$.

Theorem 6 (Dehmer [41]) *Let*

$$f(z)=a_nz^n+a_{n-1}z^{n-1}+\cdots+a_0, a_na_{n-1}\ne 0,$$

be a complex polynomial. All zeros of $f(z)$ *lie in the closed disk*

$$K\left(0,\frac{1+\phi}{2}+\frac{\sqrt{(\phi-1)^2+4M_2}}{2}\right),$$

(20)

where

$$\phi := \left|\frac{a_{n-1}}{a_n}\right| \quad and \quad M_2 := \max_{0 \le j \le n-2} \left|\frac{a_j}{a_n}\right|.$$

(21)

Besides locating the zeros of polynomials, it is often important to determine the number of positive or negative zeros of polynomials with real coefficients. In this light, we state the famous Descartes Rule of Signs, see [28], [34].

REFERENCES

1. Bonchev D, Rouvray DH. Chemical Graph Theory. 1991. Introduction and Fundamentals. Abacus Press. New York, NY, USA.

2. Dong FM, Koh KM, Teo KL. Chromatic Polynomials and Chromaticity of Graphs. 2005. World Scientific Publishing Company. New Jersey, NJ, USA.

3. Gutman I. Bonchev D, Rouvray DH, editors. Polynomials in graph theory. Chemical Graph Theory.1991. pp. 133–176. Introduction and Fundamentals, Abacus Press. New York, NY, USA.

4. Ellis-Monaghan JA, Merino C. Graph polynomials and their applications I: The Tutte polynomial. In: Dehmer M, editor. Structural Analysis of Complex Networks. Boston/Basel: Birkhäuser; 2010. pp. 219–255.

5. Gutman I , Harary F. Generalizations of the matching polynomial. Utilitas Mathematica. 1983;24:97–106.

6. Birkhoff GD. A determinant formula for the number of ways of coloring a map. The Annals of Mathematics. 1912;14:42–46.

7. Diudea MV. Nanomolecules and Nanostructures. 2010. Polynomials and Indices. Mathematical Chemistry Monographs. University of Kragujevac and Faculty of Science Kragujevac. Kragujevac, Serbia.

8. Emmert-Streib F. Algorithmic computation of knot polynomials of secondary structure elements of proteins. Journal of Computational Biology. 2006;13:1503–1512. [PubMed]

9. Wiener H. Structural determination of paraffin boiling points. Journal of the American Chemical Society. 1947;69:17–20. [PubMed]

10. Yan W, Yang BY, Yeh Y. The behavior of Wiener indices and polynomials of graphs under five graph decorations. Applied Mathematics Letters. 2007;18:290–295.

11. Zadeh HS, Hamzeh A, Ashrafi AR. Wiener-type invariants of some graph operations. Filomat.2009;23:103–113.

12. Polansky OE, Bonchev D. The Wiener number of graphs. I. General theory and changes due to graph operations. MATCH Communications in Mathematical and in Computer Chemistry. 1986;21:133–186.

13. AbuGhneim OA, Al-Ezeh H, Al-Ezeh M. The Wiener polynomial of the k-th power graph.International Journal of Mathematics and

Mathematical Sciences. 2007 Article ID 24873, doi:10.1155/2007/24873.

14. Ivanciuc O, Ivanciuc T, Diudea MV. Polynomials and spectra of molecular graphs. Roumanian Chemical Quaterly Reviews. 1999;7:41–67.

15. Gutman I, Polansky OE. Mathematical Concepts in Organic Chemistry. Springer; 1986.

16. Lovász L, Pelikán J. On the eigenvalues of trees. Periodica Mathematica Hungarica. 1973;3:175–182.

17. Randić M, Vracko M, Novic M. Diudea MV, editor. Eigenvalues as molecular descriptors.QSPR/QSAR Studies by Molecular Descriptors, Nova Publishing. 2001. pp. 93–120. Huntington, NY, USA.

18. Bonchev D. Topological order in molecules 1. Molecular branching revisited. Journal of Molecular Structure: THEOCHEM. 1995;336:137–156.

19. Todeschini R, Consonni V, Mannhold R. Handbook of Molecular Descriptors. 2002. Wiley-VCH. Weinheim, Germany.

20. Dehmer M, Sivakumar L, Varmuza K. Uniquely discriminating molecular structures using novel eigenvalue-based descriptors. MATCH Communications in Mathematical and in Computer Chemistry.2012;67:147–172.

21. Jackson B. Zeros of chromatic and ow polynomials of graphs. Journal of Geometry. 2003;76:95–109.

22. Kovše M. Counting cubes in median graphs and related problems. Analysis of Complex Networks: From Biology to Linguistic. 2009. pp. 323–350. Wiley VCH. Weinheim, Germany.

23. Woodall D. A zero-free interval for chromatic polynomials. Discrete Mathematics. 1992;101:333–341.

24. Brešar B, Klavžar S, Škrekovski R. Roots of cube polynomials of median graphs. Journal of Graph Theory. 2006;52:37–50.

25. Hosoya H. On some counting polynomials. Discrete Applied Mathematics. 1988;19:239–257.

26. Křivka P, Trinajstić N. On the distance polynomial of a graph. Applications of Mathematics.1983;28:357–363.

27. Sagan BE, Yeh YN, Zhang P. The Wiener polynomial of a graph. International Journal of Quantum Chemistry. 1996;60:959– 969.

28. Marden M. Geometry of polynomials. 1966. Mathematical Surveys of the American Mathematical Society, Vol. 3. Rhode Island, USA.

29. Rahman QI, Schmeisser G. Analytic Theory of Polynomials. 2002. Critical Points, Zeros and Extremal Properties. Clarendon Press. Oxford, UK.

30. Dehmer M. Die analytische Theorie der Polynome. 2004. Nullstellenschranken für komplexwertige Polynome. Weissensee- Verlag. Berlin, Germany.

31. Jain VK. On the zeros of polynomials II. Journal of Mathematical and Physical Sciences.1986;20:259–267.

32. J oyal A, Labelle G, Rahman QI. On the location of polynomials. Canadian Mathematical Bulletin.1967;10:53–63.

33. Mignotte M, Stefanescu D. Polynomials: An Algorithmic Approach. 1999. Discrete Mathematics and Theoretical Computer Science. Springer. Singapore.

34. Obreschkoff N. Verteilung und Berechnung der Nullstellen reeller Polynome. 1963. Hochschulbücher für Mathematik, Vol. 55. VEB Deutscher Verlag der Wissenschaften. Berlin, Germany.

35. Cvetković DM, Doob M, Sachs H. Spectra of Graphs. 1980. Theory and Application. Deutscher Verlag der Wissenschaften. Berlin, Germany.

36. Harary F. Graph Theory. 1969. Addison Wesley Publishing Company. Reading, MA, USA.

37. Lepović M, Gutman I. A collective property of trees and chemical. Journal of Chemical Information and Computer Sciences. 1998;38:823–826.

38. Montel P. Sur les modules des zéros des polynômes. Annales Scientifiques de l'École Normale Supérieure. 1923;40:1–34.

39. Fujiwara M. Über die obere Schranke des absoluten Betrages der Wurzeln einer algebraischen Gleichung. Tôhoku Mathematical Journal. 1916;10:167–171.

40. Kakeya S. On the limits of the roots of an algebraic equation with positive coefficients. Tôhoku Mathematical Journal. 1912;2:140–142.

41. Dehmer M, Mowshowitz A. Bounds on the moduli of polynomial zeros. Applied Mathematics and Computation. 2011 accepted, doi:10.1016/j.amc.2011.09.043