Stability Criteria for Mathematical Models in Ecology A Focus on Ecological Dynamics

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Abstract

Ecological systems are inherently complex and dynamic, characterized by intricate interactions among species and their environment. Mathematical models have become indispensable tools for understanding and predicting these dynamics, enabling ecologists to gain insights into the behavior of ecosystems under various conditions. One crucial aspect of ecological modelling is assessing the stability of these systems, as it determines the long-term behavior and resilience of ecosystems.We explore the various mathematical techniques and concepts used to analyze the stability of ecological models, ranging from simple one-dimensional models to complex multi-species interactions. The paper navigates through fundamental concepts such as equilibrium points, stability analysis, and bifurcation theory.introduces the foundational principles of stability analysis, focusing on equilibrium points and their classifications as stable, unstable, or semi-stable. Building upon this, we delve into the local and global stability analysis techniques, elucidating their applications and limitations. The paper then extends its exploration to bifurcation theory, discussing how qualitative changes in system behavior can occur as key parameters are varied.

Introduction

Ecological dynamics, characterized by the intricate interactions between living organisms and their environment, have long fascinated scientists seeking to decipher the complexities of natural systems. Mathematical models have emerged as indispensable tools in this endeavor, providing a structured framework to explore the underlying patterns and mechanisms driving ecological processes. This introduction provides a gateway into the realm of mathematical modeling in ecology, with a specific focus on ecological dynamics.

Ecological systems exhibit a remarkable diversity of behaviors, from the oscillations of predator-prey relationships to the delicate equilibrium of diverse species coexisting within a habitat. Mathematical models offer a means to distill these complex dynamics into quantifiable equations, enabling us to simulate, analyze, and predict ecosystem behavior. However, the effectiveness of these models hinges upon their accuracy in capturing the fundamental dynamics of ecological systems.This exploration delves into the pivotal role of mathematical models in unraveling the intricacies of ecological phenomena. By translating ecological interactions into mathematical equations, models allow us to study the influence of variables, test hypotheses, and assess the consequences of various interventions. The subsequent discussions will delve into various types of ecological models, their applications, and the challenges they pose. By delving into the diverse array of mathematical tools used to capture ecological dynamics, this review sets the stage for a comprehensive understanding of the complex web of relationships that shape ecosystems.

Mathematical Ecology

Mathematical ecology, an interdisciplinary field at the nexus of mathematics and biology, strives to unravel the intricate patterns and dynamics of ecosystems through quantitative frameworks. By translating complex ecological interactions into mathematical equations, this field provides a structured approach to model and analyze the behavior of species populations, food webs, and biodiversity. From simple differential equations to intricate agent-based simulations, mathematical ecology employs a diverse array of tools to uncover underlying ecological principles. These models aid in predicting the consequences of environmental changes, informing conservation strategies, and advancing our understanding of the delicate balance that sustains life on Earth. As ecological challenges intensify, the synergy between mathematics and ecology becomes increasingly crucial in fostering insights that guide sustainable coexistence between organisms and their environment.

Logistic Growth Model:

The logistic growth model is a fundamental equation that describes the growth of a population under limiting factors, such as resource availability.

$$\frac{dN}{dt} = rN$$

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)$$

Lotka-Volterra Predator-Prey Model:

The Lotka-Volterra model is a classic predator-prey interaction model that describes the dynamics between two interacting species, a predator and its prey.

$$\frac{dp}{dt} = \alpha P - \beta P.V$$

$$\frac{dV}{dt} = \gamma V + \delta P.V$$

$\frac{dp}{dt}$ =s the rate of change of the prey population P over time t,

$\frac{dV}{dt}$is the rate of change of the predator population V over time t,

α represents the intrinsic growth rate of the prey

β represents the predation rate.

γ represents the predator death rate,

δ represents the reproduction rate of predators per prey consumed.

These equations capture essential ecological dynamics, such as population growth, competition, and predation. However, it's important to note that real-world ecological systems are more complex and often require more sophisticated models to account for factors like spatial distribution, environmental variability, and more intricate interactions among species.

Functional Response

The concept of a functional response, or trophic function, elucidates the connection between an individual predator and its prey based on the efficiency of capturing prey. This critical notion holds significant relevance within predator-prey models, as it emulates the dynamics of predation. Let X(t) and Y(t) denote the population densities of prey and predators, respectively, at a specific time. Below, we outline fundamental types of functional responses:

Type I Functional Response: Linear Response

In this scenario, the capture rate by predators increases linearly with the density of prey, without saturation. It implies that predators consistently consume prey at a constant rate, directly proportional to prey availability.

Type II Functional Response: Saturation Response or Holling's Disk Equation

In this case, the capture efficiency initially increases with prey density, but eventually levels off due to predators reaching their maximum consumption capacity. This response acknowledges the concept of handling time, where predators need time to consume captured prey.Here thefeeding function f(X,Y) = cX, c > 0

Holling type II: A type II functional response is a hyperbolic saturating curve becausethe per capita feeding rate f depends on the number of prey in the environment. Herethe feeding function f (X,Y) = $\frac{CX}{X+a}$, C>0denotes the maximum growth rate of species and a > 0 is the half saturation constant.

Holling type III: This is a sigmoidal curve that has predators foraging inefficientlyat low prey densities. Type III functional response is similar to type II in that at highlevels of prey density, saturation occurs. Here the feeding function f(X,Y) = $\frac{CX^2}{X^{2+}a}$ a,c>0

Holling type IV: This response function describes a situation in which the predator'sper capita rate of predation decreases at sufficiently high prey densities. Here thefeeding function f(X,Y) =$\frac{CX}{Px^2+qX+a}$,a,c,p,q>0

Literature Review

Gillman, M. (2009). The natural world's complexity has ignited human curiosity for centuries, prompting the integration of mathematical approaches into the study of ecology and evolution. This introductory exploration delves into the fundamental role of mathematical models as indispensable tools for comprehending the intricate dynamics that govern ecological and evolutionary processes.Ecology and evolution, two intertwined pillars of biological science, are characterized by intricate interactions among organisms and their changing environments. Mathematical models serve as bridges between theoretical insights and empirical observations, enabling the formulation of hypotheses, prediction of outcomes, and refinement of our understanding of these intricate phenomena.Beginning with foundational concepts, the paper introduces readers to various modeling techniques that capture diverse ecological and evolutionary scenarios.

Otto, S. P., & Day, T. (2007). The dynamic interplay between biological processes and their environmental contexts has fueled the emergence of mathematical modeling as an invaluable tool for biologists exploring the realms of ecology and evolution. This guide offers a comprehensive introduction to the essential concepts and applications of mathematical modeling tailored specifically for biologists.Ecology and evolution, at the heart of biological inquiry, are inherently complex and multi-faceted fields. This guide bridges the gap between biological intuition and mathematical formalism, providing biologists with a roadmap to harnessing mathematical tools effectively. Through accessible explanations and real-world examples, readers are introduced to the diverse range of modeling techniques available.

Berec, L. (2004). The intersection of ecology and epidemiology has paved the way for a profound understanding of the intricate dynamics underlying the spread of diseases in natural populations. This abstract delves into the pivotal role of mathematical modeling in unraveling the complexities of ecological and epidemiological systems.Epidemiological processes, driven by the interplay between infectious agents, susceptible hosts, and environmental factors, are mirrored by ecological interactions in diverse ecosystems. Mathematical models serve as powerful tools that bridge these disciplines, offering a unified framework to explore disease transmission, population dynamics, and their interconnections.

Wade, M. J et al (2016) Microbial ecosystems, ubiquitous and enigmatic, harbor complex interactions that govern fundamental biogeochemical processes. This abstract delves into the diverse dimensions of mathematical modeling as a transformative tool for unraveling the intricacies of microbial ecology.Microbial communities, from the depths of soil to the expanses of oceans, shape ecosystems' functionality and resilience. Harnessing the power of mathematical models, this abstract navigates through the spectrum of interactions within microbial communities. From competition and cooperation to niche differentiation, models illuminate the dynamics that drive community assembly and succession.

Temporal and Spatial Chaotic Behavior in an Ecological System

The phenomenon of Temporal and Spatial Chaotic Behavior in an Ecological System underscores the intricate dynamics that can emerge within natural ecosystems. This phenomenon refers to the unpredictable and complex patterns of behavior exhibited by ecological systems over both time and space. The interplay of various ecological factors, such as species interactions, environmental conditions, and resource availability, can give rise to chaotic behavior that defies simple predictions.In such systems, even small changes in initial conditions can lead to drastically different outcomes over time, a characteristic commonly associated with chaos theory. Temporal chaos pertains to irregular fluctuations in population sizes and other ecological variables, while spatial chaos involves the unpredictable distribution of species and their interactions across geographic space.Understanding and characterizing Temporal and Spatial Chaotic Behavior is essential for effective ecosystem management and conservation. It highlights the challenges of making long-term predictions and underscores the need for adaptive management strategies that can accommodate the inherent unpredictability of ecological systems. Despite its complexity, studying this chaotic behavior offers valuable insights into the resilience, stability, and vulnerability of ecosystems in the face of changing environmental conditions and human interventions.

Description of Model System

In our analysis, we have explored a food chain model comprising of prey and two distinct predator levels. Within this model, the prey is designated as the primary consumer, while the predators are categorized as secondary and tertiary consumers, respectively. Specifically, the secondary consumers rely on the prey denoted as U, employing the Holling type II functional response. On the other hand, the tertiary consumer, labeled as W, feeds upon the species V, employing the Beddington-DeAngelis type functional response.

The population dynamics of the prey (U) is governed by a logistic growth framework, as encapsulated by the functional expression.F(X, Y) = $\frac{a1_X}{b1X}$where a1 and b1 are positive parameters representing the attack rate and half-saturation constant.

According to the above hypothesis, the model can be represented mathematically as follows

$$\begin{cases} \dfrac{dX}{dT} = \underbrace{rX\left(1-\dfrac{X}{K}\right)}_{\text{Logistic Growth}} - \underbrace{\dfrac{a_1(1-m)XY}{b_1+(1-m)X}}_{\text{Consumption}} - \underbrace{\dfrac{c_1EX}{l_1E+l_2X}}_{\text{Harvesting}}, \\ \dfrac{dY}{dT} = \underbrace{-dY}_{\text{Death}} + \underbrace{e_1\dfrac{a_1(1-m)XY}{b_1+(1-m)X}}_{\text{Conversional Growth}} - \underbrace{\dfrac{a_2YZ}{b_2+Y}}_{\text{Consumption}} - \underbrace{\dfrac{c_2EY}{l_3E+l_4Y}}_{\text{Harvesting}}, \\ \dfrac{dZ}{dT} = \underbrace{-cZ}_{\text{Death}} + \underbrace{e_2\dfrac{a_2YZ}{b_2+Y}}_{\text{Conversional Growth}} - \underbrace{\dfrac{c_3EZ}{l_5E+l_6Z}}_{\text{Harvesting}}, \end{cases}$$

with the initial conditions X(0) > 0, Y(0) > 0 and Z(0) > 0.

Parameters	Biological Meaning
r	Intrinsic growth rate of the prey
K	Carrying capacity
a_1	Maximum attack rate of the middle predator
a_2	Maximum attack rate of the top predator
b_1	Half-saturation constant of the prey
b_2	Half-saturation constant of the middle predator
d	Death rate of the middle predator
c	Death rate of the top predator
$c_i,\ i=1,2,3$	Catchability coefficients
E	Harvesting effort
$l_i,\ i=1,2,3,4,5,6$	Positive constants
e_1	The efficiency of the middle predator to convert the consumed prey into a new predator
e_2	The efficiency of the top predator to convert the consumed middle predator into a new predator
$m\in[0,1)$	Rate of prey refuge, remaining $(1-m)X$ non-refuged prey available for predation

Parameters used in the model and to determine the principal set of parameters, we take the following transformations

$$t = rT,\ X = Kx,\ y = \frac{a_1 Y}{rK},\ z = \frac{a_2 Z}{rK},\ w_1 = \frac{b_1}{K},\ w_2 = \frac{Ec_1}{rl_2K},\ w_3 = \frac{l_1E}{l_2K},$$
$$w_4 = \frac{d}{r},\ w_5 = \frac{e_1a_1}{r},\ w_6 = \frac{r}{a_1},\ w_7 = \frac{a_1b_2}{rK},\ w_8 = \frac{a_1c_2E}{r^2l_4K},\ w_9 = \frac{a_1l_3E}{rl_4K},$$
$$w_{10} = \frac{c}{r},\ w_{11} = \frac{e_2a_2}{r},\ w_{12} = \frac{a_2c_3E}{r^2l_6K},\ w_{13} = \frac{a_2l_5E}{rl_6K}.$$

The dimensionless system can be written as

$$\frac{dx}{dt} = x\left[(1-x) - \frac{(1-m)y}{w_1+(1-m)x} - \frac{w_2}{w_3+x}\right] = xf_1(x,\ y,\ z),$$
$$\frac{dy}{dt} = y\left[-w_4 + \frac{w_5(1-m)x}{w_1+(1-m)x} - \frac{z}{w_6(w_7+y)} - \frac{w_8}{w_9+y}\right] = yf_2(x,\ y,\ z),$$
$$\frac{dz}{dt} = z\left[-w_{10} + \frac{w_{11}y}{w_7+y} - \frac{w_{12}}{w_{13}+z}\right] = zf_3(x,\ y,\ z).$$

Analysis of Temporal System

Here, we study the model without diffusion and the transformed food chain model is given by the following system of autonomous ordinary differential equation.

$$\begin{cases} \dfrac{du}{dt} = u\Big(1-u\Big) - \dfrac{uv}{(u+w_4)} = ug_1(u,v,w), \\ \dfrac{dv}{dt} = -w_5v + \dfrac{w_6uv}{(u+w_7)} - \dfrac{vw}{(v+w_8w+w_9)} = vg_2(u,v,w), \\ \dfrac{dw}{dt} = -w_{10}w + \dfrac{w_{11}vw}{(v+w_8w+w_9)} = wg_3(u,v,w). \end{cases}$$

We have limited the non-dimensionalized system with only eight parameters.

Here ug_1, vg_2, wg_3 are continuous smooth functions

Theorem 1. Assume that condition w7 ≥ w4 holds and let Λ be the set defined by

$$\Lambda = \left[(u,v,w) \in R_+^3 : 0 \leq u \leq 1,\ 0 \leq u + \frac{v}{w_6} \leq v_c,\ 0 \leq u + \frac{v}{w_6} + \delta w \leq w_c\right],$$

Theorem 1 confirms the boundedness and dissipativeness of the system. The

proof of this theorem is given in Appendix A.

Now, to justify the feasibility and existence of our proposed food chain model, system (3.3) need to satisfy the Kolmogorov sufficient and necessary conditions (May 1973). Kolmogorov system ensures periodic oscillations or chaotic dynamics in 3D system if two subsystem of any 3D system is K-system. Let us consider the reduced subsystem of temporal model system.

$$\begin{cases} \dfrac{du}{dt} = u\left((1-u) - \dfrac{v}{u+w_4}\right) = uF(u,v), \\ \dfrac{dv}{dt} = v\left(-w_5 + \dfrac{w_6 u}{u+w_7}\right) = vG(u,v). \end{cases}$$

Within the context of two-species dynamic models, the Kolmogorov theorem plays a pivotal role in establishing the presence of either a stable limit cycle or stable equilibria. This theorem simultaneously ensures that the model adheres to biologically pertinent parameter values. Consequently, when considering the predator-prey model of ecological significance, it is reasonable to assume that the subsystem defined as (3.4) satisfies the criteria of a K-system. In essence, the characteristics of subsystem (3.4) align with those of a K-system, as delineated by the conditions.

$$0 < \frac{w_5 w_7}{w_6 - w_5} < 1,\ w_5 < w_6.$$

Moreover, the Kolmogorov system contains three positive equilibrium points. Thesustainability of these equilibria are summed as

The equilibrium E00 = (0,0) is exists with eigenvalues 1, $-w_5$. Thus, the nature ofthe eigenvalue is characterized as saddle point.

The equilibrium E10 = (1,0) always exist and is stable and eigenvalues are $-1, -w_5$

The equilibrium E20 = (u˜, v˜), where

$$\begin{cases} \tilde{u} = \dfrac{w_5 w_7}{w_6 - w_5}, \\ \tilde{v} = (1-\tilde{u})(\tilde{u}+w_4), \end{cases}$$

always exist and if

$$2\left(\frac{w_5 w_7}{w_6 - w_5}\right) + w_4 - 1 > 0,$$

Holds then E_{20} is locally asymptotically stable.

Stability of Temporal System

The stability analysis of a temporal system is a crucial endeavor in understanding its behavior over time. It involves assessing the system's response to perturbations and disturbances to determine whether it converges back to a steady state or exhibits erratic behavior. A stable temporal system maintains its equilibrium or periodic behavior in the presence of minor fluctuations, indicating its resilience to external influences. Different stability criteria, such as Lyapunov stability or asymptotic stability, are employed to quantify the system's behavior. Assessing stability provides insights into the system's long-term behavior and aids in making predictions about its future states. However, it's important to note that real-world complexities might introduce limitations to stability analyses, and some systems may exhibit sensitive dependence on initial conditions, leading to unpredictable outcomes over extended periods.

Stability and Hopf bifurcation analysis

Stability analysis involves determining the conditions under which a system's equilibrium points are stable or unstable. Stable equilibrium points signify that small perturbations around these points result in the system converging back to the equilibrium over time. Unstable equilibrium points, on the other hand, lead to divergent behavior. Various methods, such as Lyapunov stability analysis, are used to mathematically assess stability.

Hopf bifurcation refers to a critical point in a system's parameter space where a stable equilibrium loses stability, giving rise to sustained oscillatory behavior. It marks a transition from a static state to a dynamic, oscillatory regime. The analysis of Hopf bifurcation involves examining eigenvalues of the system's Jacobian matrix to determine the bifurcation point and the direction of bifurcation. This analysis sheds light on the emergence of oscillatory patterns and helps understand the transition to complex behaviors.

$$\begin{cases} g_1(u,v,w) = 1 - u - \dfrac{v}{u+w_4} = 0, \\ g_2(u,v,w) = -w_5 + \dfrac{w_6 u}{u+w_7} - \dfrac{w}{v+w_8 w+w_9} = 0, \\ g_3(u,v,w) = -w_{10} + \dfrac{w_{11} v}{v+w_8 w+w_9} = 0. \end{cases}$$

Thus, solving the equation, we get

$$\begin{cases} v^\star = (1-u^\star)(u^\star+w_4), \\ w^\star = \dfrac{(w_{11}-w_{10})v^\star - w_9 w_{10}}{w_9 w_{10}}. \end{cases}$$

For V⋆> 0 and w⋆> 0 it follows that

$$0 < u^\star < 1,\ 0 < \frac{w_9 w_{10}}{w_{11}-w_{10}} < v^\star.$$

Research problem

The focus of this study is to establish stability criteria for mathematical models within the context of ecological dynamics. Ecological systems are inherently complex and subject to various interdependent factors that influence the behaviors of species populations. Ensuring the stability of these mathematical models is of paramount importance, as it directly impacts our ability to make accurate predictions about the long-term dynamics of ecosystems.This research aims to delve into the intricacies of stability analysis, exploring different types of stability such as local, global, and asymptotic stability. By doing so, it seeks to provide a comprehensive understanding of the conditions under which ecological models exhibit stable equilibria or periodic behaviors. The study will also delve into the concept of resilience, investigating how disturbances impact stability and whether certain systems possess the capacity to recover from perturbations.In addition to stability, this research will specifically emphasize the identification and analysis of bifurcation points, particularly focusing on cases involving Hopf bifurcations. These critical points in parameter space mark the transition from steady-state behaviors to oscillatory patterns, opening up avenues for sustained cyclic dynamics within ecological systems.

Conclusion

In conclusion, the intricate tapestry of ecological dynamics finds its threads woven into the fabric of mathematical models, illuminating the underlying patterns that govern the natural world. This exploration into the realm of ecological modeling underscores the invaluable role that these models play in deciphering the complexities of ecosystems.From simple differential equations that portray predator-prey relationships to complex network models representing intricate food webs, each model offers a unique lens through which we can glimpse the mechanisms that shape ecological communities.Uncertainty in parameter estimation, the delicate balance between simplicity and accuracy, and the need to accommodate real-world intricacies pose constant hurdles. It is imperative that the use of these models be accompanied by a healthy dose of skepticism and empirical validation. Mathematical models persist as invaluable tools, enabling us to predict the consequences of human actions, understand the implications of environmental changes, and inform conservation strategies. They are bridges between theory and reality, offering a medium through which theoretical ecologists, empiricists, and policy-makers can converge to forge a deeper understanding of ecological systems.

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