AN INTERESTING DIOPHANTINE PROBLEM

K.Meena [1] ,

Former VC,

Bharathidasan University , Trichy

S.Vidhyalakshmi [2] ,

Professor,

Dept.of mathematics, SIGC,Trichy

J.Shanthi [3] ,

Lecture,

Dept.of mathematics, SIGC,Trichy,

K.Agalya [4]

M.Phil Student,

Dept.of mathematics, SIGC,Trichy

Abstract:

This paper concerns with an interesting Diophantine problem and aims at determining explicitly three distinct non-zero integers a, b, c such that the sum of any pair of them is a perfect square and twice the sum of the three integers is a perfect cube. Different methods have been considered to obtain the three required integers a, b, c. This shows that there are many triples in integers,each satisfying the considered kind of pattern among its members.

KEYWORDS: System of linear equations, Integer solutions, Cubic with four unknowns.

2010 MSC classification:11D04,11D25,11D99.

INTRODUCTION:

Number theory, called the Queen of Mathematics, is a broad and diverse part of Mathematics that developed from the study of the integers. The foundations for Number theory as a discipline were laid by the Greek mathematician Pythagoras and his disciples (known as Pythagoreans). One of the oldest branches of mathematics itself is the Diophantine equations since its origins can be found in texts of the ancient Babylonians, Chinese, Egyptians, Greeks and so on[1-6]. Diophantine problems were first introduced by Diophantus of Alexandria who studied this topic in the third century AD and he was one of the first Mathematicians to introduce symbolism to Algebra. The Greek mathematician Diophantus is renowned for his work on solving equation with rational solutions. Number Theorists relish tackling Diophantine equations with integer solutions. There is still much interest today in Diophantine problems and has been a topic of keen interest to many mathematicians worldwide because of its historical importance. They offer good applications in the fields of patterns classification, graph theory, elliptic curves, modular forms, galois representation, engineering, coding, cryptography, music, analysis of non-linear resonances in fluid mechanics and so on. The beauty of many Diophantine equations lies in the fact that they are easy to understand, yet very difficult to solve. Fermat's Last Theorem is an example to illustrate this point. If we can describe some phenomena carefully enough as a pattern, then mathematics may be able to give information about that kind of pattern. Perhaps only a mathematician would think of the empty set as a pattern- the "null patterns". In fact, Diophantine problems dominated most of the celebrated unsolved mathematical problems. Certain Diophantine problems come from physical problems or from immediate Mathematical generalizations and others come from geometry in a variety of ways. Certain Diophantine problems are neither trivial nor difficult to analyze [7-8]. In this context, one may also refer [9-12]. The above results motivated us to search for integer solutions to various other choices of Diophantine problems.

Thus, towards this end, in this communication, we search for three non-zero distinct integers such that sum of any two is a perfect square and twice their sum is a cubical integer.

METHOD OF ANALYSIS:

The Diophantine problem under consideration is to solve the system of equations

$$a+b=p^2 \qquad (1)$$

$$a+c=q^2 \qquad (2)$$

$$b+c=r^2 \qquad (3)$$

$$2(a+b+c)=s^3 \qquad (4)$$

where a,b and c represent three non-zero distinct integers.

We present below different methods of solving the system of equations (1) - (4).

METHOD I :

Consider the three integers a,b,c defined by

$$a=\frac{1}{2}(p^2+q^2-r^2), b=\frac{1}{2}(p^2-q^2+r^2), c=\frac{1}{2}(-p^2+q^2+r^2) \qquad (5)$$

It is seen that

$$a+b=p^2, a+c=q^2, b+c=r^2$$

Now, the substitution of (5) in (4) gives

$$p^2+q^2+r^2=s^3 \qquad (6)$$

Introducing the transformation

$$p = q + 2r \tag{7}$$

in (6), it is written as

$$2q^2 + 4qr + 5r^2 = s^3 \tag{8}$$

Again, substituting $q = u - v, r = v$ (9)

in (8) we get

$$2u^2 + 3v^2 = s^3 \tag{10}$$

Consider s to be

$$s = 2a_1^2 + 3b_1^2 \tag{11}$$

Substituting (11) in (10) and employing the method of factorization, define

$$\sqrt{2}u + i\sqrt{3}v = \sqrt{2}(2a_1^3 - 9a_1b_1^2) + i\sqrt{3}(6a_1^2b_1 - 3b_1^3) \tag{12}$$

Equating real and imaginary parts ,we get

$$u = 2a_1^3 - 9a_1b_1^2, v = 6a_1^2b_1 - 3b_1^3 \tag{13}$$

From (7),(9) & (13), we have

$$p = 2a_1^3 - 9a_1b_1^2 + 6a_1^2b_1 - 3b_1^3, q = 2a_1^3 - 9a_1b_1^2 - 6a_1^2b_1 + 3b_1^3, r = 6a_1^2b_1 - 3b_1^3$$

In view of (5), we have

$$a = 4a_1^6 + 63a_1^2b_1^4 - 18a_1^4b_1^2 + \frac{9}{2}b_1^6$$
$$b = 24a_1^5b_1 - 120a_1^3b_1^3 + 54a_1b_1^5 + 18a_1^4b_1^2 - 18a_1^2b_1^4 + \frac{9}{2}b_1^6$$
$$c = 18a_1^4b_1^2 - 18a_1^2b_1^4 - 24a_1^5b_1 + 120a_1^3b_1^3 - 54a_1b_1^5 + \frac{9}{2}b_1^6$$

Replacing b_1 by 2k in the above equations, we get the integer values of a, b and c to be

$$a = 4a_1^6 + 1008a_1^2k^4 - 72a_1^4k^6 + 288k^6$$
$$b = 48a_1^5k - 960a_1^3k^3 + 1728a_1k^5 + 72a_1^4k^2 - 288a_1^2k^4 + 288k^6$$
$$c = 72a_1^4k^2 - 288a_1^2k^4 - 48a_1^5k + 960a_1^3k^3 - 1728a_1k^5 + 288k^6$$

METHOD II:

Introducing the linear transformation

$$p = q + r \tag{14}$$

In (6), it is written ad

$$2q^2 + 2qr + 2r^2 = s^3 \tag{15}$$

Substituting $q = u + v, r = u - v$ (16)

in (15), we get

$$6u^2 + 2v^2 = s^3 \tag{17}$$

Consider s to be

$$s = 6a_2{}^2 + 2b_2{}^2 \tag{18}$$

Substituting (18) in (17) and employing the method of factorization, define

$$\sqrt{6}u + i\sqrt{2}v = \sqrt{6}(6a_2{}^3 - 6a_2b_2{}^2) + i\sqrt{2}(18a_2{}^2b_2 - 2b_2{}^3) \tag{19}$$

Equating real and imaginary parts ,we get

$$u = 6a_2{}^3 - 6a_2b_2{}^2, v = 18a_2{}^2b_2 - 2b_2{}^3 \tag{20}$$

From (14), (16) & (20)

$$p = 12a_2(a_2{}^2 - b_2{}^2), q = 6a_2(a_2{}^2 - b_2{}^2) + 2b_2(9a_2{}^2 - b_2{}^2), r = 6a_2(a_2{}^2 - b_2{}^2) - 2b_2(9a_2{}^2 - b_2{}^2)$$

In view of (5),we get the integer values of a, b and c to be

$$a = 12a_2(a_2{}^2 - b_2{}^2)(6a_2{}^3 - 6a_2b_2{}^2 + 18a_2{}^2b_2 - 2b_2{}^3)$$
$$b = 12a_2(a_2{}^2 - b_2{}^2)(6a_2{}^3 - 6a_2b_2{}^2 - 18a_2{}^2b_2 + 2b_2{}^3)$$
$$c = 4b_2{}^2(9a_2{}^2 - b_2{}^2)^2 - 36a_2{}^2(a_2{}^2 - b_2{}^2)^2$$

METHOD III:

Consider

$$a = 2^{2k-1}t^2 + r^2, b = 2^{2k-1}t^2 + 2^{k+1}rt, c = 2^{2k-1}t^2 - 2^{k+1}rt \tag{21}$$

where $k > 0$.

The above integers a, b, c have the property that

$$a + b = (2^k t + r)^2, a + c = (2^k t - r)^2, b + c = (2^k t)^2$$

From (21) , we have

$$2(a+b+c)=3.2^{2k}t^2+2r^2 \quad (22)$$

Now, we required the RHS of (22) to be a perfect cube. To achieve this, define

$$a=(2^{2k-1}t^2+r^2)(3.2^{2k}t^2+2r^2)^2, b=(2^{2k-1}t^2+2^{k+1}rt)(3.2^{2k}t^2+2r^2)^2,$$
$$c=(2^{2k-1}t^2-2^{k+1}rt)(3.2^{2k}t^2+2r^2)^2 \quad (23)$$

Note that $2(a+b+c)=(3.2^{2k}t^2+2r^2)^3$

Thus, (23) represent the required three non-zero distinct integers satisfying the criteria mentioned in (1) – (4).

METHOD IV:

In (22), define

$$3.2^{2k}t^2+2r^2=(2a_3^2+3.2^{2k}b_3^2)^3$$

Employing the method of factorization, define

$$\sqrt{3}.2^k t+i\sqrt{2}r=(\sqrt{2}a_3+i\sqrt{3}.2^k b_3)^3$$

After expanding, we deduce

$$t=6a_3^2b_3-3.2^{2k}b_3^3, r=2a_3^3-9.2^{2k}a_3b_3^2$$

Substituting the above values of t *and* r in (21), the required three integers a,b,c satisfying (1) – (4) are represented by

$$a=4a_3^6+2^{6k-1}9b_3^6+2^{4k}63a_3^2b_3^4-2^{2k}18a_3^4b_3^2$$
$$b=2^{2k-1}(36a_3^4b_3^2+2^{4k}9b_3^6-2^{2k}36a_3^2b_3^4)+2^{k+1}(12a_3^5b_3-2^{2k}60a_3^3b_3^3+2^{4k}27a_3b_3^5)$$
$$c=2^{2k-1}(36a_3^4b_3^2+2^{4k}9b_3^6-2^{2k}36a_3^2b_3^4)-2^{k+1}(12a_3^5b_3-2^{2k}60a_3^3b_3^3+2^{4k}27a_3b_3^5)$$

CONCLUSION:

In this paper, we have presented an interesting Diophantine problem of determining explicitly three non-zero distinct integers a,b,c such that $a+b=p^2, a+c=q^2, b+c=r^2$ and $2(a+b+c)=s^3$. The reader will now be desirous to become acquainted with the classes of indeterminate problems which Diophantus treats and his method of solution Diophantus gives only the most special solutions of all the questions which he treats and he is generally content with indicating numbers which furnish one single solution. But it must not be supposed that his method was restricted to these very special solutions. In conclusion, one may investigate several further and new explicit Diophantine problems.

ACKNOWLEDGEMENT:

The financial support from the UGC , New Delhi(F-MRP-5123/14(SERO/UGC) dated march 2014) for a part of this work is gratefully acknowledged.

REFERENCES:

[1] ***André Weil***, (1987)" Number Theory: An Approach through History, From Hammurapito to Legendre", Birkahsuser, Boston.
[2] BibhotibhusanBatta and Avadhesh Narayanan Singh, (1938) "History of Hindu Mathematics", Asia Publishing House.
[3] Boyer.C.B., (1968) "A History of mathematics", John Wiley& sons Inc., NewYork.
[4] Dickson L. E., (1952)" History of Theory of Numbers", Vol. 11, Chelsea Publishing Company, New York.
[5] Davenport.H (1999)," The higher Arithmetic: An Introduction to the Theory of Numbers"(7th ed.) Cambridge University Press.
[6] John Stilwell, (2004) "Mathematics and its History", Springer Verlag, New York.
[7] James Matteson, M.D, (1888)"A Collection of Diophantine problems with solutions", Washington, ArtemasMartin.
[8] Tituandreescu, DorinAndrica, (2002) "An Introduction to Diophantine equations" GIL Publication House.
[9]. M.A. Gopalan, S. Vidhyalakshmi, N. Thiruniraiselvi, (2015) "On Two Interesting Diophantine Problems", Impact Journal of Science and Technology, Vol-9, No-3,51-55
[10] M.A. Gopalan, S. Vidhyalakshmi, N. Thiruniraiselvi, (2015) "On Interesting Integer Pairs", IJMRME, Vol-1, ISS-1,320-323.
[11] M.A. Gopalan, S. Vidhyalakshmi, J. Shanthi, (2015) "On Interesting Diophantine Problem", IJMRME, Vol-1, ISS-1, 168-170.
[12] M.A. Gopalan, S.Vidhyalakshmi, J. Shanthi, (2015) " An Interesting Diophantine Problem", Scholar Bulletin, Vol-1, ISS-7, 166-168.