New Bounds on Domination number and Bondage number of Central and Line Graphs.

Dr.S.Velammal[1]

Velammal College of Engineering and Technology, Madurai, India

B.K.KeerthigaPriyatharsini[2]

,Research Scholar ,

Department of Mathematics,

Anna University , Chennai ,India

Abstract:

The central graph of a graph G is denoted as C(G) and is obtained from G by subdividing each edge of G exactly once and joining all other non adjacent vertices of G.In this paper, we prove that for any connected graph G,Υ(C(G)≤p-1 and for any tree T,Υ(C(T)≤p-ℓwhereℓis the number of pendant vertices of G and we characterise Connected Unicyclic graphs for which Υ(C(G)=p-1 .

Keywords

Bondage number, Central graph ,Domination number , Trees,Connected Unicyclic graphs,Bistar,Lobster,Line graph

1.Introduction

By a graph G=(V(G),E(G)) we mean a finite undirected graph without loops or multiple edges. A subset S of V is called a dominating set of G if every vertex in V-S is adjacent to some vertex in S. The domination number Υ(G) of G is the minimum cardinality among all dominating sets of G. The central graph C(G) of G is obtained from G by subdividing each edge of G exactly once and joining all other nonadjacent vertices of G

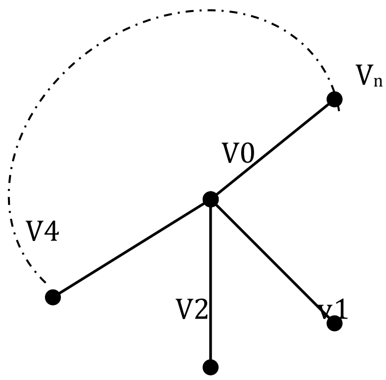
For real number x, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x and $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. We refer to [1] for the graph-theoretic terms not described here. A complete graph is a simple graph in which any two vertices are adjacent. A spanning subgraph of a graph G the subgraph obtained by edge deletions only.A connected undirected graph is called a tree when it has no cycles.*The distance d(u,v) between two vertices u and v in a connected graph G is the length of a shortest u-v path in G.A tree which yields a path when its pendant vertices are removed is called a caterpillar.A spider is a tree which has at most one vertex of degree ≥3.A bistar is the graph obtained by joining the apex vertices of two copies of star $k_{1,n}$ by an edge.A Lobster graph is a tree in which all the vertices are within distance 2 of a central path.The bondage number b(G) of a nonempty graph G is the minimum cardinality among all sets of edges E for which Υ(G-E) > Υ(G) and defined in [2].Fink et.al.[2] found the bondage number of cycles , paths and complete and complete multipartite graphs.Theorem 1[2]:The* *bondage number of the n-cycle is*

$b(C_n)=\begin{cases} 3 & if\ n \equiv 1(mod\ 3) \\ 2 & otherwise \end{cases}$

2. Main Results

Theorem2.1 $\Upsilon(C(K_{1,n}))=2$

Proof:



Let $V(K_{1,n})=\{v_0, v_1, v_2, v_3, \ldots v_n\}$ in which deg $(v_0)=n$.Then, $\{v_0, v_i\}$ where $1 \leq i \leq n$ forms a minimum dominating set of $C(K_{1,n})$.So that, $\Upsilon(C(K_{1,n}))=2$.

Theorem:2.2 $\Upsilon(C(K_p))=p-1$

Proof:

Let$V(K_p) = \{v_1, v_2, \ldots, v_p\}$and Let$x_{ij}$ be the vertex subdividingthe edge $v_i v_j$

Then clearly, $\{v_3, v_4, \ldots, v_p, x_{12}\}$ forms a minimum dominating set for $C(K_p)$

So that ,$\Upsilon(C(K_p))=p-1$

Theorem:2.3

For p≥5, $\Upsilon(C(P_k))=\begin{cases} \frac{k}{2}, k \equiv 0(mod\ 2) \\ \left\lfloor \frac{k}{2} \right\rfloor, k \equiv 1(mod\ 2) \end{cases}$

Proof:

Let $V(P_k)=\{v_1, v_2, \ldots v_k\}$

Construct , $S = \{v_1, v_3, \ldots v_k / k$ is even}.If K≡0(mod 2) ,Then S forms a dominating set with cardinality $\frac{k}{2}$.If k≡1(mod 2),Then S forms a dominating set with cardinality $\left\lfloor \frac{k}{2} \right\rfloor$

Theorem:2.4

For p≥5, $\Upsilon(C(Cp))=\left\lceil \frac{p}{2} \right\rceil$

Proof:

Let $V(C_p)=\{v_1, v_2, \ldots v_p\}$.Construct, $S = \{v_1, v_3, v_5, \ldots v_p / p$ is odd}

orms a dominating set with $\left\lceil \frac{p}{2} \right\rceil$ *vertices*.Hence $\Upsilon(C(Cp))=\left\lceil \frac{p}{2} \right\rceil$

Theorem:2.5

For p≥4 ,$\Upsilon(C(W_p)) = \left\lceil \frac{p}{2} \right\rceil+1$

Proof: Let $V(W_p)=\{v_1, v_2, \ldots v_p\}$ and v_1 be the centre of the wheel graph. Consider $V-\{v_1\} = \{v_i / 2 \leq i \leq p\}$.Let $S_1 \in V-\{v_1\}$ be the independent set of vertices with $\left\lceil \frac{p}{2} \right\rceil$-k vertices.Let $S_2 = \{x_{ij} / 2 \leq i \leq p, j=i+1$, i,j not in S_1 } be the set of subdivided vertices of these k vertices. Then, $S= S_1 \cup S_2 \cup \{v_1\}$ forms a dominating set with cardinality $|S1|+|S2|+1$ vertices.$\Upsilon(C(W_p))$ $=\left\lceil \frac{p}{2} \right\rceil - k + k + 1 = \left\lceil \frac{p}{2} \right\rceil+1$.

Theorem :2.6

$\Upsilon(C(K_{m,n}))= \min\{m,n\}+1$

Proof: $V_1=\{u_1, u_2, u_3, \ldots u_m\}$ and $V_2=\{w_1, w_2, \ldots w_n\}$ be a complete bipartition of $K_{m,n}$.

Let m≤n,As<V_2> is a complete graph in $C(K_{m,n})$, $V_1 \cup \{w_i\}$, $1 \leq i \leq n$ forms a minimum dominating set for $C(K_{m,n})$,Hence

$\Upsilon(C(K_{m,n}))= m+1$

If m≥n,$\Upsilon(C(K_{m,n}))=n+1$

Thereofore, $\Upsilon(C(K_{m,n}))=\min\{m,n\}+1$

Theorem:2.7

For any tree T with atleast two vertices of degree ≥2 ,then$\Upsilon(C(T))\leq p-\ell$ where ℓ is the number of pendant vertices of T.Equality holds if and only if either T is isomorphic to bistar or Lobster or Every non pendant vertices of T are adjacent to atleast one pendant vertices of T.

Proof: Let L be the set of all pendant vertices in T with$|L|=\ell$ and S be the set of all nonpendant vertices of T.As, In C(T) Each pendant vertex of T is adjacent to all the non pendant vertices of T ,Moreover, All the non pendant vertices of T dominates the subdivided vertices of C(T) and themselves.Then S=V-L forms a dominating set in C(T).Hence, $\Upsilon\leq p-\ell$.

Assume ϒ=p-,If diam T= 3, T is isomorphic to bistar with $\Upsilon=p-\ell$

If diam T = 4, T is isomorphic to Lobster

Suppose ,diam T > 4

Claim : Every non pendant vertex of T is adjacent to atleast one pendant vertex of T.

Suppose not, Let v be the such vertex which is not adjacent to pendant vertex in T.

As, the subdivided vertices of C(T) which are adjacent to v are dominated by the vertices of S-{v} and v is dominated by one of the vertices in S-{v} in C(T).

Hence , S-{v} forms a dominating set with $p-\ell-1$ vertices which is a contradiction to $\Upsilon=p-\ell$.Hence, Every non pendant vertex of T is adjacent to atleast one pendant vertex of T.Converse is obvious.

Theorem:2.8

For any connected graph of order p,

$\Upsilon(C(G)\leq p-1$

Proof: Let V(G) = {$v_1,v_2,\ldots\ldots v_p$}.As C(G) is connected , V(G)-{v_i} , 1≤i≤p forms a dominating set with p-1 vertices,So that , $\Upsilon(C(G)\leq p-1$

Result 2.9

If G is a connected graph in which $\Upsilon(C(G))=p-1$ then G contains at least one cycle.

Proof: Suppose, G does not contain a cycle.

Then G is isomorphic to a tree

For any tree with at least two vertices of degree ≥2,$\Upsilon(C(T))\leq p-\ell$ and $\ell\geq 2$

$\Upsilon(C(T))\leq p-2$ which is a contradiction to $\Upsilon(C(G))=p-1$

Hence G contains at least one cycle.

Theorem:2.10

Let G be connected Unicyclic graph.$\Upsilon(C(G))=p-1$ if and only if G is isomorphic to $C_3,C_4,C_{3,1}$

Proof: Let G be connected Unicyclic graph with $\Upsilon(C(G))=p-1$ and C be the unique cycle of G.If for each vertex of G has degree 2.Then G is isomorphic to C_3,C_4 since For p≥5, $\Upsilon(C(Cp))=\left\lceil\frac{p}{2}\right\rceil$.Suppose G has a vertex of degree ≥3 on C,say uWe claim that $C=C_3$,Suppose $C=C_n$, n≥4.Let V_1 be the set of $\left\lceil\frac{2n-3}{4}\right\rceil$ vertices dominating all the 2n-3 vertices of C_{2n} –{u}.Let V_2 be the set of $\frac{2p-2n}{2}$=p-n vertices dominating the remaining vertices of C(G).Clearly,{u}UV_1UV_2 is a dominating set of C(G) and hence $\Upsilon(C(G))\leq 1+\left\lceil\frac{2n-3}{4}\right\rceil$+p-n.Now, since n≥4 ,

We have n> $2+\left\lceil\frac{2n-3}{4}\right\rceil$,$\Upsilon(C(G))<1+\left\lceil\frac{2n-3}{4}\right\rceil$+p-2-

$\left\lceil\frac{2n-3}{4}\right\rceil$<p-1 which is a contradiction .Thus C=C_3

Now suppose C_3 has two vertices u,v with deg u ≥3 and deg v ≥3 and u, v be the vertices of C(G) lying on C.Now, Let the vertex of C_6not adjacent to u be u_1 {u,v,u_1} dominates atleast p+5 vertices of C(G).

Let V_1 be the collection of p-5 vertices dominating the remaining vertices of C(G). Then {u,v,u_1}UV_1 forms a dominating set of C(G) of cardinality 3+p-5=p-2 which is a contradiction. Hence C_3 has exactly one vertex u with deg u ≥3

Suppose there exists a pendant vertex v not on C_3 such that d(u,v)=r≥2

Let x_{ij} be the subdividing vertex of v_i and v_j ,v_i ≠u,v_j ≠u on C

Then u and x_{ij} dominate all the vertices of C_6 and at least one subdivided vertex which are adjacent to u. Let V_1 be the collection of p-4 vertices dominating remaining vertices of C(G) except the subdivided vertices ,not on C which are adjacent to u. Clearly {u ,x_{ij}}UV_1 forms a dominating set of C(G) of cardinality 2+p-4=p-2 which is a contradiction. Hence there exists a pendant vertex v not on C_3 such that d(u, v)=1.

Suppose there exist two pendant vertices v_1 ,v_2 not on C_3 such that d(u,v_1)=1 & d(u,v_2)=1. Then clearly,{x_{12},u,v_i} forms a dominating set of C(G) of cardinality 3 which is a contradiction. Since p≥5,p-1≥4,Hence there exists only one pendant vertex v not on C_3 such that d(u,v)=1.Hence G is isomorphic to C_3,C_4,$C_{3,1}$.

The converse is obvious.

Theorem: 2.11

Let G be connected (p,q) graph.

ThenΥ(C(G))≤p-1 ,Equality holds if and only if G is isomorphic to P_3 , W_5,C_4,K_p or G contains at most one pendant vertex with four vertices.

Proof: Let G be connected graph with Υ(C(G))=p-1

Claim(i) G is isomorphic to P_3 , W_5,C_4,K_p or G contains at most one pendant vertex with four vertices. By Result 2.9, G can not be tree

case(i) Suppose all the vertices having degree p-1 ,Then G is complete graph

Since Υ(C(K_p))=p-1,case(ii)Suppose all the vertices having some degree except p-1 and deg u = r (say),

Claim : r≤2,Suppose 3≤r<p-1

Then G is a regular graph of order p of degree r and q edges. C(G) has p+q vertices since q=$\frac{rp}{2}$,Hence C(G) has

p+q=p+$\frac{rp}{2}$=$\frac{2p+rp}{2}$=$\frac{(2+r)p}{2}$ vertices

Since 3≤r<p-1, One vertex u of C(G) dominates p-1 vertices in C(G)

Then remaining $\frac{(2+r)p}{2}$-(p-1)-1 =$\frac{rp}{2}$ vertices are dominated by p-r vertices

Hence, C(G) has dominating set with cardinality 1+p-r,Since 3≤r<p-1

Υ(C(G))< p-1 which is a contradiction to Υ=p-1,Hence r≤2

If r=1, G is isomorphic to K_2, If r=2,G is isomorphic to C_p

Since Υ=p-1,G is isomorphic to C_3,C_4

case(iii) Suppose for some vertex of G having degree<p-1,Claim p≤5,

Suppose not, Let T be the spanning tree of G

Let u be the vertex of T having maximum degree Δ(say)

Then u dominates Δ+(p-Δ)-1=p-1 vertices in C(T)

Since C(T) has 2p-1 vertices, Remaining 2p-1-(p-1)-1=2p-1-p+1-1=p-1 vertices are dominated by at most $\left\lceil\frac{p}{2}\right\rceil$ vertices in C(T),ϒ(C(T))≤$\left\lceil\frac{p}{2}\right\rceil + 1$, But ϒ(C(G))≤ ϒ(C(T))≤$\left\lceil\frac{p}{2}\right\rceil + 1$,Since p≥6 , ϒ<p-1 which is a contradiction to ϒ=p-1

Hence p≤5, If p=4 and ϒ=p-1,G contains atmost one pendant vertex

If p=5,G is isomorphic to W_5.Conversely, Assume G is isomorphic to P_3 , W_5,C_4,K_p or G contains at most one pendant vertex with four vertices.

Then ϒ=p-1 , this is obvious

Some New Results on Bondage number of Line graphs

First we determine ϒ(L(G) for some special classes of graphs.

Lemma 1.

The domination number of the path of order k is ϒ(L(P_k))=$\left\lceil\frac{k}{3}\right\rceil$ for k≥3

Proposition 1.

The bondage number of the k-cycle of

b(L(C_k))=$\begin{cases} 3 & if\ n \equiv 1(mod\ 3) \\ 2 & otherwise \end{cases}$

Proof.

By theorem 1[2] ,it is obvious

Proposition 2.

For k≥3,b(L(P_k))=$\begin{cases} 2 & k \equiv 2(mod\ 3) \\ 1 & otherwise \end{cases}$

Proof

Trivially b(L(P_k))≥1

If k≡2(mod 3), the dismissal of one edge from L(P_k) yields a graph A which is to be composed of two paths P_1 and P_2.

If P_1 has order k_1 and P_2 has order k_2, then either k_1≡0(mod 3),k_2≡1(mod 3)or k_1≡2(mod 3),k_2≡2(mod 3)

In the first mentioned of the two cases,ϒ(A)=ϒ(P_1)+ϒ(P_2)=$\left\lceil\frac{k_1}{3}\right\rceil+\left\lceil\frac{k_2}{3}\right\rceil=\frac{k_1}{3}+\frac{k_2+2}{3}=\left\lceil\frac{k}{3}\right\rceil$

In the latter case,

ϒ(A)=ϒ(P_1)+ϒ(P_2)=$\left\lceil\frac{k_1}{3}\right\rceil+\left\lceil\frac{k_2}{3}\right\rceil=\frac{k_1+1}{3}+\frac{k_2+1}{3}=\left\lceil\frac{k}{3}\right\rceil$. By Lemma 1, In these two cases, we conclude that , b(L(P_k))≥2

To obtain the upper bounds that , by trichotomy , will produces the desired equalities of our theorem's statement , we consider three cases,

Case(i) Suppose that k≡2(mod 3),The subgraph A gained by removing two edges from L(P_K). ϒ(A)=2+ϒ(P_{k-3})=2+$\left\lceil\frac{k-3}{3}\right\rceil>\left\lceil\frac{k}{3}\right\rceil$

The subgraph A gained by removing one edge form L(P_k)

Subcase(i) Suppose that k≡0(mod 3)

ϒ(A)=1+ϒ(P_{k-2})=1+$\left\lceil\frac{k-2}{3}\right\rceil>\left\lceil\frac{k}{3}\right\rceil$

Subcase(ii) k≡1(mod 3),

ϒ(A)=1+$\left\lceil\frac{k-2}{3}\right\rceil>\left\lceil\frac{k}{3}\right\rceil$

By Lemma 1., with the earlier inequality we conclude that b(L(P_k))=$\begin{cases} 2 & k \equiv 2(mod\ 3) \\ 1 & otherwise \end{cases}$

Proposition 3.

For $K_{1,n}$,b(L($K_{1,n}$))=$\left\lceil\frac{n}{2}\right\rceil$

Proof.

Since the Line Graph of star graph is complete graph

Hence by Proposition 1[2] the result holds.

3. References

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