Pythagorean Triangle with Hypotaneous minus $4\left(\frac{Area}{Perimeter}\right) = (4k^2+2)\alpha^2$

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ABSTRACT:

Patterns of Pythagorean triangles in each of which Hypotaneous minus $4\left(\frac{Area}{Perimeter}\right) = (4k^2+2)\alpha^2$. Also, a few diophantine quadruples with suitable property are constructed through the linear combination among the generators of the Pythagorean triangle. The Pythagorean numbers play a significant role in the theory of higher arithmetic as they come in the majority of indeterminate problems; had a marvelous effect on a credulous people and always occupy a remarkable position due to unquestioned historical importance. The method of obtaining three non-zero integers *x*,*y* and *z* under certain relations satisfying the relation $x^2+y^2=z^2$ has been a matter of interest to various Mathematicians [1],[14]-[18]. In [2]-[13], special Pythagorean problems are studied. In this communication, we search for patterns of Pythagorean triangles wherein each of which hypotaneous minus $4\left(\frac{Area}{Perimeter}\right) = (4k^2+2)\alpha^2$. Also, a few diophantine quadruples with suitable property are constructed through the linear combination among the generators of the Pythagorean triangle.

Keywords: Area/perimeter, Pythagorean triangle, Square integer, Diophantine quadruple.
Mathematical Subject Classification: 11D09,11D99

Introduction:

The method of obtaining non-zero integers x, y and z under certain conditions satisfying the relation $x^2 + y^2 = z^2$ has been a matter of interest to various Mathematicians [1],[14]-[18]. In [2]-[13], special Pythagorean problems are studied. In this communication, we search for patterns of Pythagorean triangles wherein each of which hypotaneous minus $4\left(\frac{Area}{Perimeter}\right) = (4k^2 + 2)\alpha^2$. Also, a few diophantine quadruples with suitable property are constructed through the linear combination among the generators of the Pythagorean triangle.

METHOD OF ANALYSIS

The most cited solution of the Pythagorean equation,

$$x^2 + y^2 = z^2 \tag{1}$$

is represented by

$$x = 2pq; y = p^2 - q^2; z = p^2 + q^2 \tag{2}$$

Denoting the Area and Perimeter of the Pythagorean triangle by A and P respectively, the assumption

$$z - 4\frac{A}{P} = (4k^2 + 2)\alpha^2, (k, \alpha > 0) \tag{3}$$

leads to the equation

$$(p-q)^2 + 2q^2 = (4k^2 + 2)\alpha^2 \tag{4}$$

Consider $\alpha = a^2 + 2b^2$ (5)

Using (5) in (4), it is written in the factorizable form as,

$$\left[(p-q) + i\sqrt{2}q\right]\left[(p-q) - i\sqrt{2}q\right] = (2k + i\sqrt{2})(2k - i\sqrt{2})(a+ib)^2(a-ib)^2$$

which is equivalent to the system of two equations

$$\left[(p-q) + i\sqrt{2}q\right] = (2k + i\sqrt{2})(a+ib)^2$$

$$\left[(p-q) - i\sqrt{2}q\right] = (2k - i\sqrt{2})(a-ib)^2$$

Equating the real and imaginary parts in either of the above equations and performing an algebra, we have

$$\left.\begin{aligned} p &= (a^2 - 2b^2)(2k+1) + 4ab(k-1) \\ q &= a^2 - 2b^2 + 4abk \end{aligned}\right\} \tag{6}$$

The condition $p > q > 0$ gives $k(a^2 - 2b^2) > 2ab$ (7)

For simplicity and clear understanding, we search for Pythagorean triangles satisfying the relation (3) when $k = 1, 2$

Case 1:

Let $k = 1$ Then(3) becomes $(p-q)^2 + 2q^2 = 6\alpha^2$ (8)

From (6) we have

$$\left.\begin{aligned} p(a,b) &= 3a^2 - 6b^2 \\ q(a,b) &= a^2 - 2b^2 + 4ab \end{aligned}\right\} \qquad (9)$$

Substituting the values of p and q in (2) we get the values of $x, y \,\&\, z$.

$$\left.\begin{aligned} x(a,b) &= 6a^4 - 24a^2b^2 + 24a^3b - 48ab^3 + 24b^4 \\ y(a,b) &= 8a^4 - 48a^2b^2 - 8a^3b + 16ab^3 + 32b^4 \\ z(a,b) &= 10a^4 - 24a^2b^2 + 8a^3b - 16ab^3 + 40b^4 \end{aligned}\right\} \qquad (10)$$

Thus, equations (10) represent the sides of the Pythagorean triangle satisfying (8) provided that $(a-b)^2 \succ 3b^2$ where $a = 3n + k - 1 \,\&\, b = n$

we present below different methods of solving (8) and thus, obtain different choices of Pythagorean triangles satisfying (3) when $k = 1$

Remark:

In (8) one may write 6 as

$$6 = \frac{(2 + i19\sqrt{2})(2 - i19\sqrt{2})}{121}$$

Following the procedure as above a different Pythagorean triangle satisfying (3) is obtained.

Method I:

Equation (8) can be rewritten as

$$6\alpha^2 - 2q^2 = (p-q)^2 \times 1 \qquad (11)$$

Write 1 as $1 = \dfrac{(3\sqrt{6} + 5\sqrt{2})(3\sqrt{6} - 5\sqrt{2})}{4}$ (12)

Substituting (12) in (11), we get

$$\frac{(3\sqrt{6} + 5\sqrt{2})(3\sqrt{6} - 5\sqrt{2})}{4}(\sqrt{6}a + \sqrt{2}b)^2(\sqrt{6}a - \sqrt{2}b)^2 = (\sqrt{6}\alpha + \sqrt{2}q)(\sqrt{6}\alpha - \sqrt{2}q)$$

Equating the rational and irrational parts, we get the values of $\alpha \,\&\, q$

$$\alpha = \alpha(a,b) = 9a^2 + 3b^2 + 10ab$$

$$\left.\begin{aligned} p &= p(a,b) = 21a^2 + 3b^2 + 18ab \\ q &= q(a,b) = 15a^2 + 5b^2 + 18ab \end{aligned}\right\} \qquad (13)$$

Substituting the values of $p \,\&\, q$ in (2) we get the values of $x, y \,\&\, z$.

$$\left.\begin{aligned} x(a,b) &= 630a^4 + 948a^2b^2 + 1296a^3b + 288ab^3 + 30b^4 \\ y(a,b) &= 216a^4 - 24a^2b^2 + 216a^3b - 72ab^3 - 16b^4 \\ z(a,b) &= 666a^4 + 924a^2b^2 + 1296a^3b + 288ab^3 + 34b^4 \end{aligned}\right\}$$

The above solution represent the sides of the Pythagorean triangle satisfying (8) provided that $3a^2 \succ b^2$

Remark:

One may consider the representations for 1 as given below

$$1=\begin{cases}\dfrac{(9\sqrt{6}+\sqrt{2})(9\sqrt{6}-\sqrt{2})}{484}, \\ \dfrac{(17\sqrt{6}+25\sqrt{2})(17\sqrt{6}-25\sqrt{2})}{484}, \\ \dfrac{(19\sqrt{6}+29\sqrt{2})(19\sqrt{6}-29\sqrt{2})}{484}, \\ \dfrac{(\sqrt{6}+\sqrt{2})(\sqrt{6}-\sqrt{2})}{4}, \\ \dfrac{(11\sqrt{6}+19\sqrt{2})(11\sqrt{6}-19\sqrt{2})}{4}, \\ \dfrac{(11\sqrt{6}+5\sqrt{2})(11\sqrt{6}-5\sqrt{2})}{484}, \\ \dfrac{(17\sqrt{6}+23\sqrt{2})(17\sqrt{6}-23\sqrt{2})}{484}, \end{cases}$$

Following the above procedure and obtaining the suitable condition, we get the different sides of the Pythagorean triangle such that $Hyp-4\dfrac{A}{P}=6\alpha^2$

Method II:

Equation (8) can be rewritten as,

$$6\alpha^2-(p-q)^2=2q^2 \qquad (14)$$

Assume $q(a,b)=6a^2-b^2;\ a,b\neq 0$ (15)

Write 2 as $2=\dfrac{(3\sqrt{6}+\sqrt{2})(3\sqrt{6}-\sqrt{2})}{25}$ (16)

Applying the procedure presented in Pattern II we get the values of $p,q\,\&\,\alpha$ as

$$\left.\begin{aligned} p(a,b)&=\frac{1}{5}\left[42a^2-3b^2+36ab\right] \\ q(a,b)&=6a^2-2b^2 \\ \alpha(a,b)&=\frac{1}{5}\left[18a^2+2b^2+4ab\right]\end{aligned}\right\} \qquad (17)$$

Since our aim is to find the integer solution, substitute $a=5A\,\&\,b=5B$**.,** Hence the values of $x,y\,\&\,z$ are

$$\left.\begin{aligned} x(A,B)&=750\left[84A^4 20A^2B^2+72A^3B-12AB^3+B^4\right] \\ y(A,B)&=21600A^4+33600A^2B^2+75600A^3B-5400AB^3-400B^4 \\ z(A,B)&=66600A^4+18600A^2B^2+75600A^3B-5400AB^3+850B^4\end{aligned}\right\}$$

The values of $x,y\,\&\,z$ represent the sides of the Pythagorean triangle with the condition that $(B-6A)^2 \prec 50A^2$ such that $Hyp - 4\frac{A}{P} = 6\alpha^2$

Remark:

Instead of (13) one may consider the following representation for 2

$$2 = \begin{cases} (\sqrt{6}+2)(\sqrt{6}-2), (9\sqrt{6}+22)(9\sqrt{6}-22), \\ \frac{(11\sqrt{6}+26)(11\sqrt{6}-26)}{25}, \frac{(11\sqrt{6}+2)(11\sqrt{6}-2)}{361}, \\ \frac{(17\sqrt{6}+22)(17\sqrt{6}-22)}{625}, \frac{(17\sqrt{6}+26)(17\sqrt{6}-26)}{529}, \\ \frac{(19\sqrt{6}+22)(11\sqrt{6}-22)}{841}, \frac{(19\sqrt{6}+46)(19\sqrt{6}-46)}{25}, \end{cases}$$

Repeating the procedure presented in Pattern III and finding the suitable condition, we get the different sides of the Pythagorean triangle such that $Hyp - 4\frac{A}{P} = 6\alpha^2$

Method III:

Equation (4) can be written in the ratio form as,

$$\frac{p-q+2\alpha}{\alpha-q} = \frac{2(\alpha+q)}{p-q-2\alpha} = \frac{A}{B}$$

This is equivalent to the following two equations,

$$B(p-q) + (2B-A)\alpha + Aq = 0$$
$$-A(p-q) + 2(B+A)\alpha + 2Bq = 0$$

Applying the method of cross multiplication, and simplifying we have

$$p-q = 4B^2 - 2A^2 - 4AB$$

$$\left.\begin{aligned} p &= 6B^2 - 3A^2 \\ q &= 2B^2 - A^2 + 4AB \end{aligned}\right\} \qquad (18)$$

$$\alpha = -(A^2 + 2B^2)$$

Hence the values of $x, y \,\&\, z$ are

$$\left.\begin{aligned} x(A,B) &= 24B^4 - 24B^2A^2 + 48AB^3 - 24A^3B + 6A^4 \\ y(A,B) &= 32B^4 - 48B^2A^2 - 16AB^3 + 8A^3B + 8A^4 \\ z(A,B) &= 40B^4 - 24B^2A^2 + 16AB^3 - 8A^3B + 10A^4 \end{aligned}\right\}$$

Thus, the values of $x, y \,\&\, z$ represent the sides of the Pythagorean triangle satisfying (8) provided that $2B^2 - A^2 \succ 2AB$

3.Remarkable Observations:

By considering suitable linear combination among the generators p, q are may construct diophantine 3-tuples with suitable property. A few illustrations are given below for each Methods:

Table.1: Diophantine quadruple generated from (9)

P	Q	Property	Diophantine Quadruple
$3\left[p(a,1)+6\right]$	$p(a,1)+q(a,1)+8$	$9a^2$	$(9a^2, 4a^2+4a, 25a^2+10a, 400a^4$ $+560a^3+236a^2+28a)$
$p(a,1)$	$a\left[3q(a,1)-p(a,1)\right]$	6^2	$(3a^2-6, 12a^2, 27a^2+6, 108a^6-288a^4$ $228a^2-48)$
$p(a,1)+2$	$2\left[p(a,1)-q(a,1)+4\right]$	$25a^2$	$(a^2+4a, 4a^2-4a, 9a^2+6a,$ $\frac{1}{25}144a^4+528a^3+412a^2-84a)$

Table.2: Diophantine quadruple generated from (13)

P	Q	Property	Diophantine Quadruple
$6\left[p(a,1)-q(a,1)+2\right]$	$\frac{p(a,1)+q(a,1)-8}{36}$	$9a^2$	$(36a^2, a^2+a, 49a^2+7a, 784a^4$ $+896a^3+284a^2+16a)$
	$\frac{p(a,1)+q(a,1)-8}{6}$	$9a^2$	$(6a^2, 6a^2+6a, 24a^2+12a, 384a^4+576a^3$ $264a^2+36a)$
	$\frac{p(a,1)+q(a,1)-8}{9}$	a^2	$(a^2+4a, 4a^2-4a, 9a^2+6a,$ $144a^4+240a^3+124a^2+20a)$

Table.3: Diophantine quadruple generated from (17)

P	Q	Property	Diophantine Quadruple
$7\left[\frac{p(A,1)+15}{30}\right]$	$\frac{q(A,1)+25}{6}$	$225A^2$	$(49A^2+42A, 25A^2, 144A^2+72A,$ $3136A^4+4256A^3+1780A^2+228A)$
$\frac{p(A,1)+15}{6}$		$5625A^2$	$(1225A^2+1050A, 25A^2, 1600A^2+1200A, \frac{1}{5625}(196000000A^4$ $+315000000A^3+158062500A^2+25312500A))$
$\frac{p(A,1)+15}{5}$		$8100A^2$	$(1764A^2+1512A, 25A^2, 2209A^2+1692A, \frac{1}{8100}(389667600A^4$ $+632469600A^3+32059800A^2+51904800A))$

Table.4: Diophantine quadruple generated from (18)

P	Q	Property	Diophantine Quadruple
$\frac{p(1,B)+3}{3}$	$\frac{q(1,B)+1}{2}$	$4B^2$	$(4B^2, B^2+2B, 9B^2+6B, 36B^4+96B^3+76B^2+16B)$
$\frac{p(1,B)+3}{3}$		$9B^2$	$(9B^2, B^2+2B, 16B^2+8B, 64B^4+160B^3+116B^2+20B)$
$\frac{p(1,B)+3}{6}$		B^2	$(B^2, B^2+2B, 4B^2+4B, 16B^4+48B^3+44B^2+12B)$

Case 2:

Let $k=2$ Then (3) becomes $(p-q)^2+2q^2=18\alpha^2$ (19)

From (6) and (7) we have

$p(a,b)=5a^2-10b^2+4ab$\

$q(a,b)=a^2-2b^2+8ab$

and $a^2>b(a+2b)$

The above inequality is satisfied when $b=n, a=m+2n; m,n=1,2,3...$

Therfore,

$$\left.\begin{aligned} p&=p(m,n)=5m^2+18n^2+24mn \\ q&=q(m,n)=m^2+18n^2+12mn \end{aligned}\right\} \quad (20)$$

In view of (2) the sides of the Pythagorean triangle satisfying (3) are given by,

$x(m,n)=10m^4+168m^3n+792m^2n^2+1296mn^3+648n^4$

$y(m,n)=24m^4+216m^3n+576m^2n^2+432mn^3$

$z(m,n)=26m^4+264m^3n+936m^2n^2+1296mn^3+648n^4$

Observation:

From the generators p,q gives by (20), one may construct Diophantine quadruple with suitable property.

Illustration :

Let From the generators $P=p(m,1)-q(m,1)$ & $Q=q(m,1)-12m-18$. The quadruple (P,Q,R,S) is a Diophantine quadruple with property $D(9m^2)$ where

$R=9m^2+18m$, $S=16m^4+80m^3+124m^2+60m$

Proof:

$$PQ+9m^2=(2m^2+3m)^2$$
$$PR+9m^2=(6m^2+15m)^2$$
$$PR+9m^2=(6m^2+15m)^2$$
$$PS+9m^2=(8m^3+32m^2+27m)^2$$
$$QS+9m^2=(4m^3+10m^2+3m)^2$$
$$RS+9m^2=(12m^3+42m^2+33m)^2$$

Thus the quadruple (P,Q,R,S) is a Diophantine quadruple with property $D(9m^2)$

Note:1

(19) is written as $(p-q)^2=18\alpha^2-2q^2$ (21)

Introducing the linear transformations

$$\alpha=X+2T,q=X+18T \quad (22)$$

in (21) leads to $(p-q)^2=16X^2-576T^2$

which is satisfied by

$$(p-q)=2rs,24T=r^2-s^2,4X=r^2+s^2 \quad (23)$$

Since our is to find an integer solution substitute in $r=24R,s=24S$ in (23) and therefore we get the values of α & q as

$$\left.\begin{aligned}\alpha&=192R^2+96S^2\\ q&=576R^2-288S^2\end{aligned}\right\} \quad (24)$$

Also , $p=576R^2+1152RS-288S^2$

and $2R^2>S^2$, substituting the values of the generators p,q in (2), the sides of the Pythagorean triangle satisfying (3) is given by

$$x(R,S)=663552R^4+1327104R^3S-663552R^2S^2-663552RS^3+165888S^4$$
$$y(R,S)=RS\left[1327104R^2+1327104RS-663552S^2\right]$$
$$z(R,S)=663552R^4+1327104R^3S+663552R^2S^2-663552RS^3+165888S^4$$

Observation:

From the generators p,q one may construct Diophantine quadruple with suitable property.

Illustration :

Let From the generators $A=\dfrac{p(R^2,1)-q(R^2,1)}{2}$ & $B=q(R,1)+432$. The quadruple (A,B,C,D) is a Diophantine quadruple with property $D(5184)$ where

$C=2304R^2+288$, $D=589284R^6+221184R^4+25344R^2+864$

Proof:

$$AB+5184=(576R^2+72)^2$$
$$AC+5184=(1152R^2+72)^2$$

$$AD + 5184 = (18432R^4 + 3456R^2 + 72)^2$$
$$BC + 5184 = (1152R^2 + 216)^2$$
$$BD + 5184 = (18432R^4 + 5760R^2 + 360)^2$$
$$CD + 5184 = (36864R^4 + 9216R^2 + 504)^2$$

Thus the quadruple (A, B, C, D) is a Diophantine quadruple with property $D(5184)$

Conclusion

To conclude one may consider the Pythagorean triangle under consideration,

References:

[1] Dickson.L.E, 1952, "History of theory of numbers",vol.2,Chelsea Publishing Company,NewYork,**.**

[2] Gopalan M.A. and Leelavathi.S, 2007, "Pythagorean triangle with 2(Area/Perimeter) as a cubic integer", *Bulletin of Pure and Applied Sciences*,vol.27 E(2), pp. 197-200.

[3] Gopalan M.A. and Janaki, G, (2008, "a") "Pythagorean triangle with Area/Perimeter as a special polygonal number",*Bulletin of Pure and Applied Sciences*,vol.27 E(2), pp. 393-402.

[4] Gopalan M.A. and Devibala. S, (2008, "b") "Pythagorean triangle with triangular number as a leg", *Impact J.Sci.Tech*. vol.2(4), pp.195-199.

[5] Gopalan M.A. and Janaki.G, (2008, "c") "Pythagorean triangle with nasty number as a leg",*Journal of Applied Mathematical Analysis and Applications*",vol.4(1-2),pp.13-17.

[6] Gopalan M.A. and Janaki. G, (2008, "d") "Pythagorean triangle with perimeter as a pentagonal number",*AntarcticaJ.Math.*,vol.5(2), pp.15-18.

[7] Gopalan M.A. and Gnanam .A, (2010, "a") " Pythagorean triangles and special polygonal numbers",*International Journal of Mathematical Sciences* vol.9(1-2), pp.211-215.

[8] Gopalan M.A. and Sangeetha.G, (2010, "b") "Pythagoream triangles with perimeter as triangular number",*The Global Journal of Applied Mathematics and Mathematical Sciences*,vol.3(1-2),2010,pp.93-97.

[9] Gopalan M.A. and Sivakami .B, 2012, "Pythagorean Triangle with Hypotenuse minus 2(Area/Perimeter) as a squareinteger",*Archimedes J.Math.*,vol.2(2), pp.153-166.

[10] Gopalan M.A. and Geetha V., (2013, "a") "Pythagorean triangle with Area/Perimeter as a special polygonal number",*International Refereed Journal of Engineering and Science*,vol.2(7), pp.28-34.

[11] Gopalan M.A., ManjuSomanath and Geetha .K, (2013, "b") "Pythagorean triangle with Area/Perimeter as a special polygonal number",*IOSR-JM*,vol.7(3),pp52-62.

[12] Gopalan M.A., ManjuSomanath and Sangeetha .V, (2013, "c") "Pythagorean triangles and pentagonal number",*Cayley J.Math,*,vol.2(2),2013,pp.151-156.

[13] Gopalan M.A., ManjuSomanath and Sangeetha.V, (2013, "d") "Pythagorean triangles and special pyramidal numbers", *IOSR-JM*, vol.7(4), 2013,pp.21-22.

[14] Mordell L.J., 1969 ,"Diophantine Equations",AcademicPress,New York

[15] Nagell T., 1988 ,"Introduction to Number Theory",Plencem,New York.

[16] Smith D.E. , 1953 "History of Mathematics",vol.1 and 2,Dover Publications, New York.

[17] Telang S.G. , 1996 "Number Theory",Tata McGraw-Hill Publishing Company,New Delhi.

[18] Thomas Koshy, 2005 "Elementary Number Theory with Applications",Academic Press.