Representation Theorem for Distributional Fourier-Mellin Transform

V.D.Sharma

HOD, Mathematics Department,
Arts, Commerce and Science College,
Amravati, 444606 (M.S.) India

Abstract:

Signals can be abstracted and expressed by the functions in mathematics. It is well known that the Fourier and Mellin transform used in the signal processing due to its enormous properties. The Fourier-Mellin transform as object oriented video coding and digital image water marking induced a renewal of interest for scenes representation.

In this paper the representation theorem for the distributional Fourier-Mellin transform is proved.

Keywords:

Fourier Transform, Mellin Transform, Testing function Space, Fourier-Mellin Transform .

1. Introduction:

In the late 1970's, the optical research community introduced the Fourier-Mellin Transformed for the pattern recognition and it was latter used in the digital signal and image processing. Several sets of rotation and scale invariant features based on Fourier-Mellin Transform modulus have been designed. The Fourier-Mellin transform applied in the field of image analysis such as object orient coding, content-based retrieval and digital water marking induced a renewal of interest for the scenes representation [6].

Signals can be abstracted and expressed by the functions in mathematics. The problems of reconstruction of signals and images with finite support from their Fourier transform phases to relevant to various applications of information processing, such as signal and image coding and phase contrast x-ray computed tomography [7] and due to its scale invariant property Mellin transform is also used in signal processing [8].

Fourier and Mellin transforms are related mathematically and due to shift invariant of Fourier transform and scale invariant property Mellin transform the Fourier-Mellin transform is powerful tool in signal processing [7]. Mellin transform a kind non-linear transformation, is widely used for its scale invariance property, the Fourier and Mellin transforms were generalized into multidimensional types and fractional types. All these transformations provide us alternative way to analyze the spectra of different signals [8]. The representation theorem state that every abstract structure with certain properties is isomorphic to a concrete structure

The distributional Fourier-Mellin transform is defined as follows-

For $f(t,x) \in FM_{a,b,\alpha}^{\beta}$ where $FM_{a,b,\alpha}^{*\beta}$ is the dual space and $a < Rep < b,\ s > 0$.
$FM\{f(t,x)\} = F(s,p) = \langle f(t,x), e^{-ist}\ x^{p-1}\rangle$
Where, for each $x(0 < x < \infty)$, $t(0 < t < \infty)$.

The conventional Fourier-Mellin Transform is defined as-

$$FM\{f(t,x)\} = F(s,p) = \int_0^\infty \int_0^\infty f(t,x)\, e^{-ist}\ x^{p-1}\, dt\, dx$$

In the present work Representation theorem for the distributional Fourier-Mellin Transform is proved.

2 Testing Function Space:

2.1 $FM^{\beta}_{a,b,\alpha}$ Space:

The testing function space $FM^{\beta}_{a,b,\alpha}$-

Let I be the open set $R_+ \times R_+$ and E_+ denotes the class of infinitely differentiable function defined on I, the space-

$$FM^{\beta}_{a,b,\alpha} = \{\emptyset \in E_+ / \ \gamma_{a,b,k,q,l} = \sup_{I_1} \left| t^k\ \xi_{a,b}(x) x^{q+1} D^l_t D^q_x\ \emptyset_{\Delta s,\Delta t}(t,x) \right| < CA^k k^{k\alpha} B^q q^{q\beta}\}$$

Where, $A.B$ & C are the constants depends on testing function space $\emptyset$.

2.2 The space $FM_{a,b,\alpha}$:

Let I be the open set $R_+ \times R_+$ and E_+ denotes the class of infinitely differentiable function defined on I, the space-

$$FM_{a,b,\alpha} = \{\emptyset \in E_+ / \ \rho_{a,b,k,q,l} = \sup_{I_1} \left| t^k\ \xi_{a,b}(x) x^{q+1} D^l_t D^q_x\ \emptyset_{\Delta s,\Delta t}(t,x) \right| < CA^k k^{k\alpha}\}$$

Where, The constant C & A are depend on testing function space $\emptyset$.

3. Representation Theorem

Theorem:

Let f be an arbitrary element of $FM^{*}_{a,b,\alpha}(\Omega)$ and $\emptyset$ be an element of $\mathcal{D}(\Omega)$ the space of infinitely with compact support onΩ. Then there exist a bounded measurable functions $g_{m,n}(t,x)$ defined over Ω such that-

$$\langle f,\emptyset\rangle = \langle \sum_{m=0}^{r+1} \sum_{n=0}^{v+1} (-1)^{m+n} t^k\ \xi_{a,b}(x) x^{q+1} D^l_t D^q_x\ g_{m,n}(t,x)\,, \emptyset(t,x)\rangle$$

Where k is a fixed real number and r, v are appropriate non-negative integers satisfying $m \le r+1,\ n \le v+1$.

Proof:

Let $\{v_{a,b,k,q,l}\}^{\infty}_{k,l,q=0}$ be the sequence of seminorms. Let f and $\emptyset$ be the arbitrary element of $FM^{*}_{a,b,\alpha}(\Omega)$ and $\mathcal{D}(\Omega)$ respectively. Then by boundedness property of generalized function Zemanian, we have for an appropriate constant c and non-negative integer r and v satisfying$|l| \le r, |q| \le v$.

$$|\langle f,\emptyset\rangle| \le c \max_{\substack{|l|\le r\\ |q|\le v}} v_{a,b,k,q,l}\,\emptyset \le c \max_{\substack{|l|\le r\\ |q|\le v}} \sup_{t,x} \left| t^k \xi_{a,b}(x) x^{q+1} D_t^l D_x^q\, \emptyset(t,x)\right|$$

$$\le c \max_{\substack{|l|\le r\\ |q|\le v}} \sup_{t,x}\ t^k \xi_{a,b}(x) x^{q+1} \left| \sum_{m=0}^{l} \sum_{n=0}^{q} B_n\, D_t^m D_x^n\, \emptyset(t,x)\right|$$

$$\le c^{'} \max_{\substack{|l|\le r\\ |q|\le v}} \sup_{t,x} t^k \xi_{a,b}(x) x^{q+1} \max_{\substack{m\le l\\ n\le q}} D_t^m D_x^n\, \emptyset(t,x)$$

Where $c^{'}$ is constant which depends only on m, n and hencel, q. So,

$$|\langle f,\emptyset\rangle| \le c^{''} \max_{\substack{|m|\le r\\ |n|\le v}} \sup_{t,x} t^k \xi_{a,b}(x) x^{q+1} D_t^l D_x^q\, \emptyset(t,x) \quad \text{--------} \qquad (3.1)$$

Now let us set-

$$\emptyset_{r,v}(t,x) = t^k \xi_{a,b}(x) x^{q+1} \emptyset(t,x), \qquad m \le r, n \le v \quad \text{--------} \qquad (3.2)$$

Then clearly $\emptyset_{r,v}(t,x) \in \mathcal{D}(\Omega)$. On differentiating (2) partially w.r.t. x and t we get-

$$D_t D_x \emptyset = x^{-(q+1)} t^{-k} \{\frac{k}{t}[\frac{q+1}{x}\xi_{a,b}(x) - D_x\xi]\,\emptyset_{r,v}(t,x) + [D_x\xi - \frac{q+1}{x}\xi_{a,b}(x)]\,D_t\emptyset_{r,v} - \frac{k}{t}\xi_{a,b}(x) D_x\emptyset_{r,v} + \xi_{a,b}(x) D_t D_x \emptyset_{r,v}\}$$

Let suppose that inΩ,sup $\emptyset$ = sup $\emptyset_{r,v} = [A,B]$. Then since $x^{-(q+1)} t^{-k} \xi_{a,b}(x) > 0$.

$$|D_t D_x \emptyset| = x^{-(q+1)} t^{-k} \{\frac{|k|}{A}[\frac{|q+1|}{B}\left|\xi_{a,b}\right| - |D_B\xi|]\,|\emptyset_{r,v}| + [|D_B\xi| - \frac{|q+1|}{B}\left|\xi_{a,b}\right|]\,|D_t\emptyset_{r,v}| - \frac{|k|}{A}\left|\xi_{a,b}\right|\,|D_x\emptyset_{r,v}| + \left|\xi_{a,b}\right|\,|D_t D_x\emptyset_{r,v}|\}$$

$$\le C^{'''} x^{-(q+1)} t^{-k} \{|\emptyset_{r,v}| + |D_t\emptyset_{r,v}| + |D_x\emptyset_{r,v}| + |D_t D_x \emptyset_{r,v}|\}$$

Where,

$$C^{'''} = \max\{\frac{|k|}{A}[\frac{|q+1|}{B}\left|\xi_{a,b}\right| - |D_B\xi|], [|D_B\xi| - \frac{|q+1|}{B}\left|\xi_{a,b}\right|], \frac{|k|}{A}\left|\xi_{a,b}\right|, \left|\xi_{a,b}\right|\}$$

If C^{iv} is constant which depends on k and q then-

$$|D_t D_x \emptyset| \le C^{iv} x^{-(q+1)} t^{-k} |D_t D_x \emptyset_{r,v}|$$

Hence by induction we prove that in Ω, for obvious constant C^v

$$|D_t^m D_x^n \emptyset| \le C^v x^{-(q+1)} t^{-k} \sum_{\substack{c\le m\\ d\le n}} |D_t^c D_x^d \emptyset_{r,v}|$$

Substituting this into (3.1)-

$$|\langle f,\emptyset\rangle| \le c^{vi} \max_{\substack{|m|\le r\\ |n|\le v}} \sup_{t,x} |D_t^c D_x^d \emptyset_{r,v}| \quad , \quad \text{where } c \le m, d \le n \qquad (3.3)$$

Now we can write-

$\sup_{t,x}\left|\emptyset_{r,v}(t,x)\right| \le \sup_{t,x}\left|\int_t^\infty \int_x^\infty D_t D_x \emptyset_{r,v}(t,x)\ dt\ dx\right|$

$$\le \sup_{t,x}\left\|D_t D_x \emptyset_{r,v}(t,x)\right\|_{L^{'}\times L^{'}}$$

Let the product space $L^{'} \times L^{'}$ be denoted by $(L^{'})^2$. We consider linear one to one mapping $\tau: \emptyset \to \{D_t D_x \emptyset_{r,v}\}_{\substack{m\le r+1\\ n\le v+1}}$ of $\mathcal{D}(\Omega)$ into$(L^{'})^2$. In view of (4) we see that the linear functional $\tau: \emptyset_{r,v} \to \langle f, \emptyset\rangle$ is continuous on $\tau\mathcal{D}(\Omega)$ for the topology induced $(L^{'})$. Hence, by Hahn- Banach theorem, it can be continuous linear functional in the whole of $(L^{'})^2$. But the dual of $(L^{'})^2$ is isomorphic with $(L^\infty)^2$, Therefore, there exist two L^∞ functions $g_{m.n}(m \le r+1, n \le v+1)$ such that -

$\langle f, \emptyset\rangle = \sum_{\substack{m\le r+1\\ n\le v+1}} \langle g_{m.n}, D_t^m D_x^n \emptyset_{r,v}\rangle$

By (2) we have-

$\langle f, \emptyset\rangle = \sum_{\substack{m\le r+1\\ n\le v+1}} \langle g_{m.n}, D_t^m D_x^n t^k \xi_{a,b}(x) x^{q+1} \emptyset(t,x)\rangle$

Now by using property of differentiation of distribution and property of multiplication of distribution by an infinitely smooth function

$$\langle f, \emptyset\rangle = \sum_{\substack{m\le r+1\\ n\le v+1}} \langle (-1)^{m+n} t^k \xi_{a,b}(x) x^{q+1}\ g_{m.n}, \emptyset(t,x)\rangle$$

Where, $g_{m.n}(t,x)$ are bounded measurable function defined over $\Omega = (0,\infty)$. Therefore,

$\therefore\ f = \sum_{\substack{m\le r+1\\ n\le v+1}} (-1)^{m+n} t^k \xi_{a,b}(x) x^{q+1}\ g_{m.n}(t,x)$

Hence proved.

Conclusion:

In the present paper the representation theorem for the Fourier-mellin transform is proved.

References:

[1] Debnath Loknath :-" Integral Transform and their application", Second edition, London, New York, 2007.

[2] Zemanian A.H.:-"Generalized integral transformation", Inter science publisher, New York, 1968.

[3] Pathak R.S.: "A Representation theorem for class of Stieltjes Transformable Generalized function", Indian Mathematical Society, (1974), pp 392-342.

[4] Zemanian A.H.: "Distribution Theory and Transform analysis", Mc Graw-Hill, New York,1965.

[5] Igore Lyuboshenko and Alexander Akhmetshin:-"Stable signal and image reconstruction from noisy Fourier transform phase", IEEE transaction on signal processing, Vol, 47, No. 1, Jan. 1999.

[6] Stephan Demode and Fouzi Ghorbel: "Robust efficient Fourier-Mellin transform approximation for gray level image reconstruction and complete invariant description", computer vision and image understanding, 83, 57-78(2001).

[7] Sharma V.D. and Dolas P.D.: "Inversion formula for Two dimensional generalized Fourier-Mellin transform and its application", Int, Jr, of Advanced Scientific and Technical Research, Vol.2, Issue 1, Dec. 2011.

[8] Yonggong Peng, Yixian Wang, Xiangu Zao, Linuo Gong: "Properties of franctinal Mellin transfom", Advances in information Sciences and Service Sciences (AISS), Vol.5, No.5, March-13.