THE CONNECTED COMPLEMENT DOMINATION IN GRAPHS

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ABSTRACT

A subset D ⊆ V(G) of a graph G = (V, E) is said to be **connected complement domination set** of G if D is connected domination set of G and dominating set of $\bar{G}$. The **connected complement domination number** is the minimum cardinality taken over all connected complement dominating sets of G and is denoted by $\boldsymbol{\gamma_{\overline{cc}}(G)}$. In this paper, $\gamma_{\overline{cc}}(G)$ are obtained for some standard graphs.

Keywords: Connected domination number, Global domination number, Connected complement domination number.

1. INTRODUCTION

In this paper G = (V, E) a finite, simple, connected and undirected graph has p-vertices and q-edges. Terms not defined here are used in the sense of Harary [2]. The complement of $\boldsymbol{\bar{G}}$ of G is the graph with vertex set V in which two vertices are adjacent iff they are not adjacent in G. Degree of a vertex v is denoted by d(v), the maximum(minimum)degree of a graph G is denoted by $\Delta(G)(\delta(G))$. A vertex v is said to be isolated vertex if d(v) = 0.

A set D of vertices of a graph G = (V, E) is a *dominating set* of G if every vertex in V-D is adjacent to some vertex in D. The *domination number* $\boldsymbol{\gamma(G)}$ of G is the minimum cardinality of all dominating sets of G. This concept was introduced by Ore in [1].

The concept of connected domination number was introduced by E. Sampathkumar and Walikar in [3]. A set D ⊆ V(G) is said to be *connected dominating set*, if the induced subgraph <D> is connected. The *connected domination number* $\boldsymbol{\gamma_c(G)}$ of a connected graph G is the minimum cardinality of a connected dominating set of G.

In [4], E.Sampathkumar introduced the concept of global domination number as follows: A set D ⊆ V(G) is said to be *global dominating set*, if D is a dominating set of G and $\bar{G}$.The *global domination number* $\boldsymbol{\gamma_g(G)}$ is the minimum cardinality of a global dominating set of G.

In this paper, we introduced the connected complement domination by combining the concept of connected domination and global domination for a connected graph. The characteristic was studied and the exact value of the connected complement domination was found for some standard graphs and bounds. The characteristics was studied and the exact.

2. MAIN RESULTS

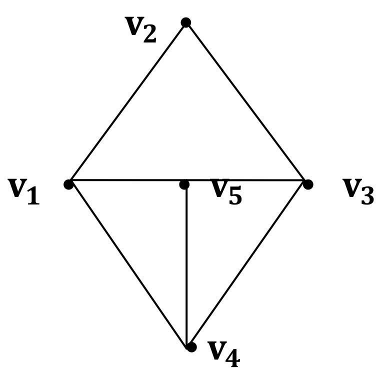
2.1. CONNECTED COMPLEMENT DOMINATION NUMBER

Definition 2.1.1

A vertex set D ⊆ V(G) of a graph G = (V, E) is said to be **connected complement dominating set**(ccd-set) of G if D is connected domination set of G and dominating set of $\bar{G}$. The **connected complement domination number** is the minimum cardinality taken over all connected complement dominating sets of G and is denoted by $\boldsymbol{\gamma_{\overline{cc}}(G)}$. A ccd-set of G with minimum cardinality is denoted by $\gamma_{\overline{cc}}$ - set of G.

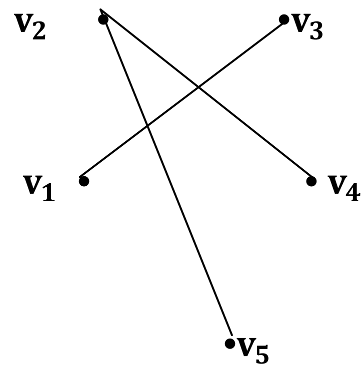
Throughout this chapter, we assume that G is connected.

Example 2.1.1 (a)



G

Figure 2.1

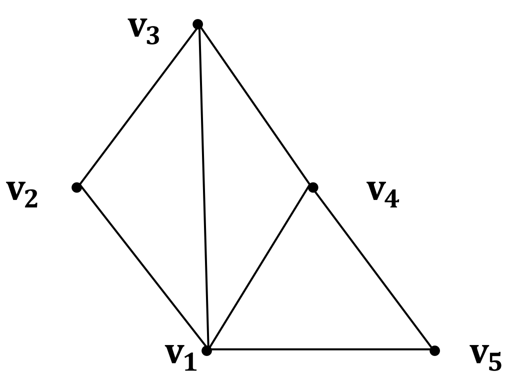


$\overline{G}$

Figure 2.2

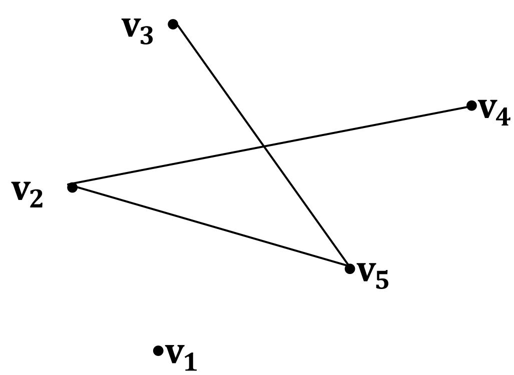
For the graph G in figure 2.1, the vertex set D = $\{v_1, v_2\}$ is a $\gamma_{\overline{cc}}$-set and hence $\gamma_{\overline{cc}}(G)$ = 2.

Example 2.1.1(b)



G

Figure 2.3



$\overline{G}$

Figure 2.4

For the graph G in figure 2.3, the vertex set D = $\{v_1, v_2, v_3\}$ is a $\gamma_{\overline{cc}}$-set and hence $\gamma_{\overline{cc}}(G)$ = 3.

2.2 BOUNDS AND RESULTS ON $\gamma_{\overline{cc}}(G)$

The following theorem gives the relation between domination, connected domination and connected complement domination for a graph G.

Theorem: 2.2.1

For any graph G, $\gamma(G) \leq \gamma_c(G) \leq \gamma_{\overline{cc}}(G)$

Proof:

Since every connected dominating set is a dominating of G,

We have $\gamma(G) \leq \gamma_c(G)$ …(1)

Also, every connected complement dominating set is a connected dominating set of G and hence

$\gamma_c(G) \leq \gamma_{\overline{cc}}(G)$ …(2)

From (1) & (2), the result follows. ■

For example the graph G in figure 2.1, the vertex set D=$\{v_1, v_2\}$ is a $\gamma_{\overline{cc}}$-set.

So, $\gamma_{\overline{cc}}(G)$ = 2. Since <D> is connected, $\gamma_c(G)$ = 2.

Therefore, we have $\gamma_c(G) = \gamma_{\overline{cc}}(G)$.

From the graph G in figure 2.3, the vertex set D = $\{v_1, v_2, v_3\}$ is a $\gamma_{\overline{cc}}$-set.

So, $\gamma_{\overline{cc}}(G)$ = 3. D_1 = $\{v_1, v_2\}$ is a γ_c - set and hence $\gamma_c(G)$ = 2.

Therefore, we have $\gamma_c(G) < \gamma_{\overline{cc}}(G)$.

The exact values of connected complement domination number $\gamma_{\overline{cc}}(G)$ for some standard graphs are given below.

Theorem: 2.2.2

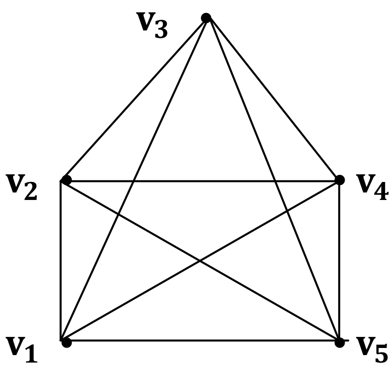
For the complete graph K_n, $\gamma_{\overline{cc}}(K_n) = n$, $n \geq 2$.

Proof:

Let V (G) = $\{v_1, v_2, v_3, \ldots, v_n\}$ be the vertex set of G.

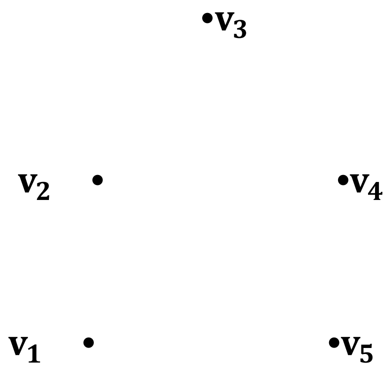
Since in $\overline{K}_n$, all the vertices are isolated and hence $\gamma_{\overline{cc}}$ – set must contain all the vertices of K_n. Therefore, $\gamma_{\overline{cc}}(K_n) = n$. ■

Example



$\mathbf{K_5}$

Figure 2.5



$\overline{K}_5$

Figure 2.6

For the graph K_n in figure 2.5, the vertex set D = $\{v_1, v_2, v_3, v_4, v_5\}$ is a $\gamma_{\overline{cc}}$-set and hence $\gamma_{\overline{cc}}(K_n)$ = 6.

Theorem: 2.2.3

For the complete bipartite graph $K_{m,n}$, $\gamma_{\overline{cc}}(K_{m,n}) = 2$ for m, n ≥ 2.

Proof:

Let G be a complete bipartite graph with at least 4 vertices.

Let $V(G)$ = $\{u_1, u_n, u_3,\ldots, u_m, v_1, v_2, v_3,\ldots, v_n\}$ be the vertex set of G.

Then the set S = $\{u_i, v_i\}$forms a connected complement set of G.

Hence $\gamma_{\overline{cc}}(G) \leq |S| = 2$ …(1)

Let S be the $\gamma_{\overline{cc}}$-set of G. Since $\bar{G}$ contains two complete components, then for the domination of $\bar{G}$, S must contain atleast one vertex from each component in $\bar{G}$. Hence S has at least 2 vertices.

$\gamma_{\overline{cc}}(G) = |S| \geq 2$ … (2)

Therefore, from equations (1) and (2),

We have $\gamma_{\overline{cc}}(G) = 2$ for m, n ≥ 2. ■

Example

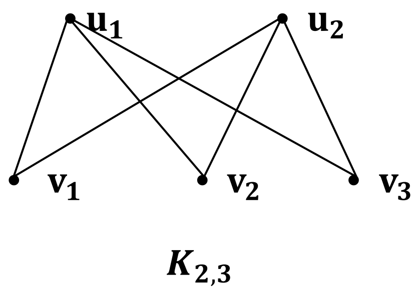


Figure 2.7

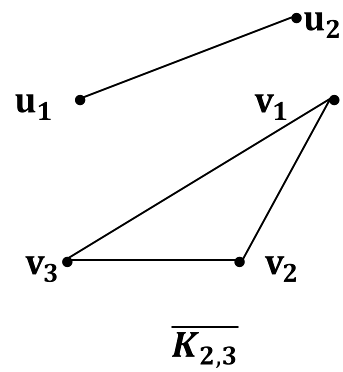


Figure 2.8

For the graph $K_{m,n}$ in figure 2.7, the vertex set D = $\{u_1, v_2\}$ is a $\gamma_{\overline{cc}}$-set and hence $\gamma_{\overline{cc}}(K_{m,n})$ = 2.

Theorem: 2.2.4

For the star graph $K_{1,n}$, $\gamma_{\overline{cc}}(K_{1,n}) = 2$ for $n \geq 2$.

Proof:

Let G be a star graph $K_{1,n}$ with atleast 3 vertices and V (G) = $\{u, v_1, v_2, v_3, \ldots, v_n\}$ be the vertex set of G. Let $u \in V(G)$ has the maximum degree in G and v_i be any vertex adjacent to u in G. Then the set $S = \{u, v_i\}$ forms a connected complement set of G.

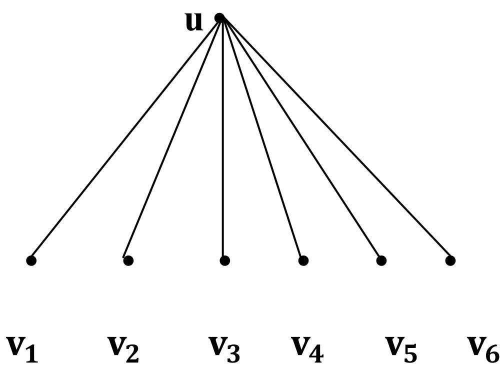
Hence $\gamma_{\overline{cc}}(G) \leq |S| = 2$ …(1)

Let S be the $\gamma_{\overline{cc}}$ - set of G. The dominating set in $\bar{G}$ must contain the isolated vertex and one maximum degree vertex in $\bar{G}$. Hence the connected complement set has at least 2 vertices.

$\gamma_{\overline{cc}}(G) = |S| \geq 2$ … (2)

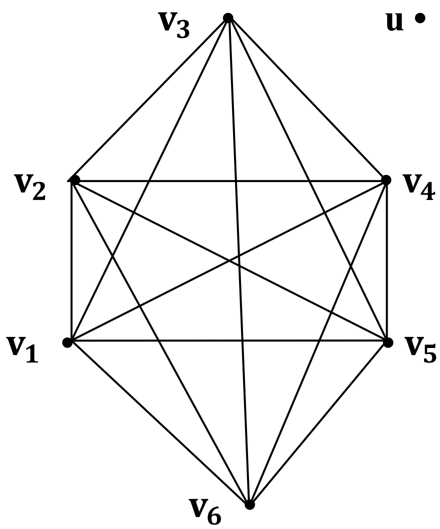
Then, the result follows from (1) and (2). ■

Example



$K_{1,6}$

Figure 2.9



$\overline{K_{1,6}}$

Figure 2.10

For the graph $K_{1,6}$ in figure 2.9, the vertex set D = $\{u, v_2\}$ is a $\gamma_{\overline{cc}}$-set and hence $\gamma_{\overline{cc}}(K_{1,6}) = 2$.

Theorem: 3.2.5

For the wheel graph W_n, $\gamma_{\overline{cc}}(W_n) = \begin{cases} 3 & , n = 2 \\ 4 & , n = 3 \\ 3 & , n \geq 4 \end{cases}$

Proof:

Let G be a wheel graph W_n with atleast 3 vertices and

V (G) = $\{u, v_1, v_2, v_3, \ldots, v_n\}$ such that d(u) = $\Delta(G)$ and d(v_i) = 2.

Case (i): $n = 2, 3$

Since the vertex set $V(G)$ itself is a minimum connected complement set of G and

Hence $\gamma_{\overline{cc}}(G) = |V(G)|$ which proves the result.

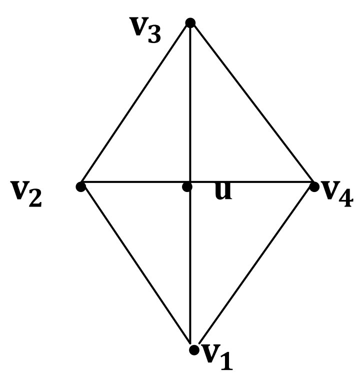
Case (ii): $n \geq 4$

Let $u \in V(G)$ has the maximum degree in G, and v_i, v_{i+1} be any two adjacent vertices in $V(G)$. Then the set $S = \{u, v_i, v_{i+1}\}$ forms a connected complement set of G.

Hence $\gamma_{\overline{cc}}(G) \leq |S| = 3$ …(1)

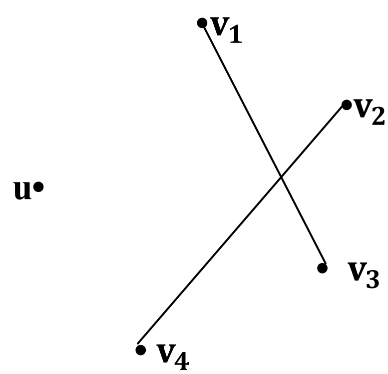
Let S be the $\gamma_{\bar{c}}$ - set of G. The dominating set in $\bar{G}$ must contain the isolated vertex and atleast any two non-adjacent vertices in $\bar{G}$ and hence the connected complement set has atleast 3 vertices. $\gamma_{\overline{cc}}(G) = |S| \geq 3$ … (2) Then, the result follows from (1) and (2). ∎

Example



W_4

Figure 2.11



$\overline{W}_4$

Figure 2.12

For the graph W_n in figure 2.11, the vertex set D = $\{u, v_2, v_3\}$ is a $\gamma_{\overline{cc}}$-set and hence $\gamma_{\overline{cc}}(W_n)$ = 3.

Theorem: 2.2.6

For the fan graph $F_{m,n}$, $\gamma_{\overline{cc}}(F_{m,n}) = 3$ for $m \geq 1, n \geq 2$.

Proof:

Let G be a fan graph $F_{m,n}$ with atleast 3 vertices and

V (G)= $\{u_1, u_n, u_3, \ldots. u_m, v_1, v_2, v_3, \ldots, v_n\}$ be a vertex set of G.

Let $u_1 \in V(G)$ has the maximum degree in G, and v_i, v_{i+1} be any two adjacent vertices in $V(G)$.

Then the set $S = \{u_1, v_i, v_{i+1}\}$ forms a connected complement set of G.

Hence $\gamma_{\overline{cc}}(G) \leq |S| = 3$ …(1)

Let S be the $\gamma_{\overline{cc}}$ - set of G. The dominating set in $\bar{G}$ must contain the isolated vertex and two non-adjacent vertices in $\bar{G}$.

Hence the connected complement set has at least 3 vertices

$$\gamma_{\overline{cc}}(G) = |S| \geq 3 \quad \ldots (2)$$

Then, the result follows from (1) and (2). ■

Example

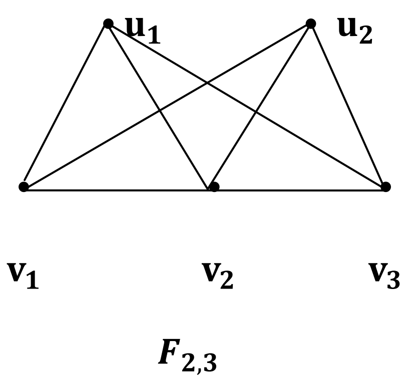


Figure 2.13

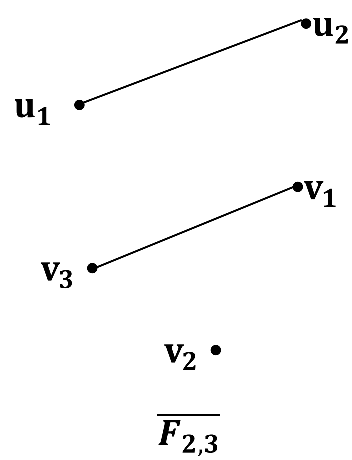


Figure 2.14

For the graph $F_{2,3}$ in figure 2.13, the vertex set D = { u_1, v_1, v_2 } is a

$\gamma_{\overline{cc}}$-set and hence $\gamma_{\overline{cc}}(F_{2,3})$ = 3.

Theorem: 2.2.7

For the banana tree $B_{n,n}$, $\gamma_{\overline{cc}}(B_{n,n}) = 2$ for n ≥ 2.

Proof:

Let G be a banana tree with at least 6 vertices and

V (G)= { $u, v, u_1, u_2, u_3, \ldots. u_n, v_1, v_2, v_3, \ldots. v_n$ }be a vertex set of G.

Let $u, v \in V(G)$ such that d(u) = d(v) = $\Delta(G)$ then the set $S = \{u, v\}$ forms a connected complement set of G and

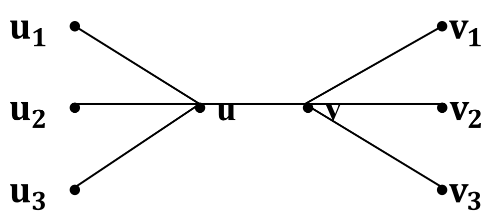
Hence $\gamma_{\overline{cc}}(G) \leq |S| = 2$ …(1)

Let S be the $\gamma_{\overline{cc}}$-set of G. For the domination of $\bar{G}$, S must contain atleast two minimum degree non-adjacent vertices in $\bar{G}$ and Hence $\gamma_{\overline{cc}}(G) = |S| \geq 2$ …(2)

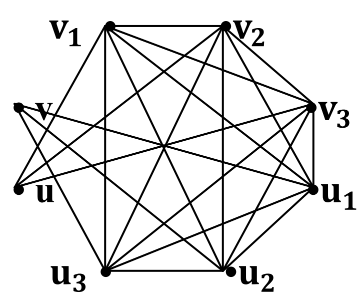
Therefore, from equations (1) and (2),

We have $\gamma_{\overline{cc}}(G) = 2$ for n ≥ 2. ■

Example



***$B_{3,3}$* (Figure 2.15)**



$\overline{B_{3,3}}$ (Figure 2.16)

For the graph $B_{3,3}$ in figure 2.15, the vertex set D = { u, v} is a $\gamma_{\overline{cc}}$ - set and hence $\gamma_{\overline{cc}}(B_{3,3})$ = 2.

Theorem: 2.2.8

For the book graph B_n, $\gamma_{\overline{cc}}(B_n) = 2$ for n ≥ 2.

Proof:

Let G be a book graph B_n with at least 6 vertices and

V(G) = { $u, v, u_1, u_2, u_3, \ldots, u_n, v_1, v_2, v_3, \ldots, v_n$ }be a vertex set of G.

Let $u, v \in V(G)$ has the maximum degree in G and of same degree then the set $S = \{u, v\}$ forms a connected complement set of G and hence

$$\gamma_{\overline{cc}}(G) \leq |S| = 2 \quad \ldots(1)$$

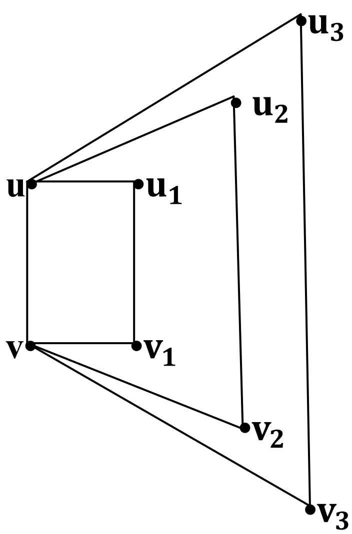
Let S be the $\gamma_{\overline{cc}}$ - set of G. The dominating set in $\bar{G}$ must contain atleast two minimum degree non-adjacent vertices in $\bar{G}$ and hence

$$\gamma_{\overline{cc}}(G) = |S| \geq 2 \qquad \ldots (2)$$

Therefore, from equations (1) and (2), We have $\gamma_{\overline{cc}}(G) = 2$ for n ≥ 2. ■

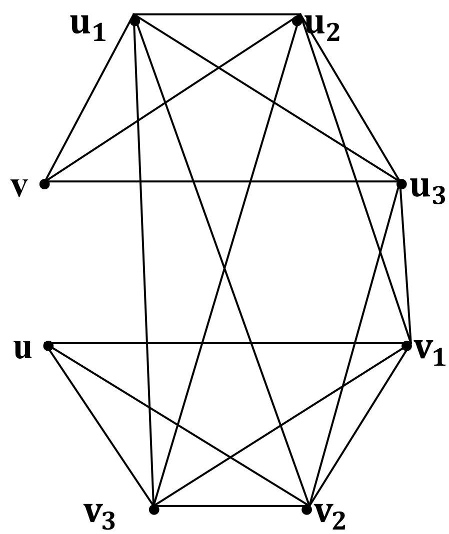
Example

For the graph B_3 in figure 2.17, the vertex set D = { u, v} is a $\gamma_{\overline{cc}}$-set and hence $\gamma_{\overline{cc}}(B_3)$ = 2.



B_3

Figure 2.17



$\overline{B_3}$

Figure 2.18

Theorem: 2.2.9

For n-barbell graph, $\gamma_{\overline{cc}}(G) = 2$, for n ≥ 3.

Proof:

Let G be a n-barbell graph with at least 6 vertices and

V(G) = { $u, v, u_1, u_2, u_3, \ldots, u_n, v_1, v_2, v_3, \ldots, v_n$ } be a vertex set of G.

Let $u, v \in V(G)$ has the maximum degree in G and of same degree then the set $S = \{u, v\}$ forms a connected complement set of G and hence

$$\gamma_{\overline{cc}}(G) \leq |S| = 2 \qquad \ldots (1)$$

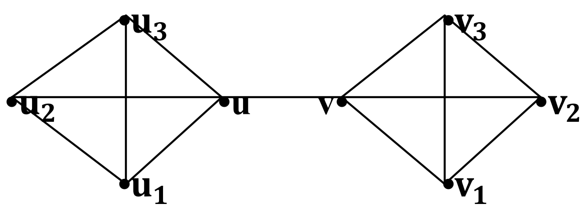
Let S be the $\gamma_{\overline{cc}}$ - set of G. The dominating set in $\bar{G}$ must contain atleast two minimum degree non-adjacent vertices in $\bar{G}$ and

Hence $\gamma_{\overline{cc}}(G) = |S| \geq 2$ … (2)

Therefore, from equations (1) and (2),

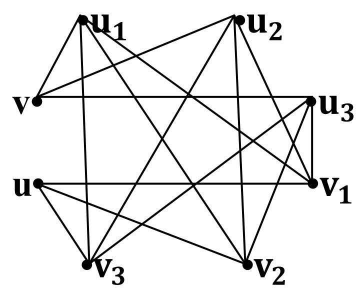
We have $\gamma_{\overline{cc}}(G) = 2$ for n ≥ 3. ■

Example



G

Figure 2.19



$\bar{G}$

Figure 2.20

For the graph G in figure 2.19, the vertex set D = {u,v} is a $\gamma_{\overline{cc}}$ - set and hence $\gamma_{\overline{cc}}(G)$ = 2.

Theorem: 2.2.10

For the friendship graph, $\gamma_{\overline{cc}}(C_3^m) = 3$, for m ≥ 2.

Proof:

Let G be a friendship graph with at least 5 vertices and V (G) = {$u, v_1, v_2, v_3,….,v_n$} be a vertex set of G.

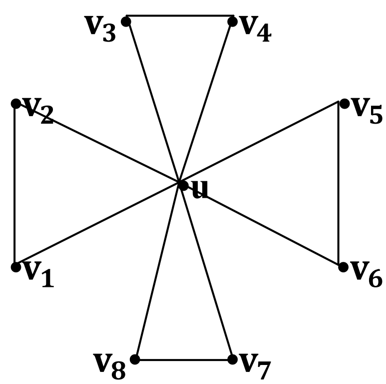
Let $u \in V(G)$ has the maximum degree in G, and v_i, v_{i+1} be any two adjacent vertices in $V(G)$. Then the set $S = \{u, v_i, v_{i+1}\}$ forms a connected complement set of G.

Hence $\gamma_{\overline{cc}}(G) \leq |S| = 3$ …(1)

Let S be the $\gamma_{\overline{cc}}$-set of G. The dominating set in $\bar{G}$ must contain the isolated vertex and atleast any two non-adjacent vertices in $\bar{G}$ and hence the connected complement set has atleast 3 vertices. Hence $\gamma_{\overline{cc}}(G) = |S| \geq 3$ … (2)

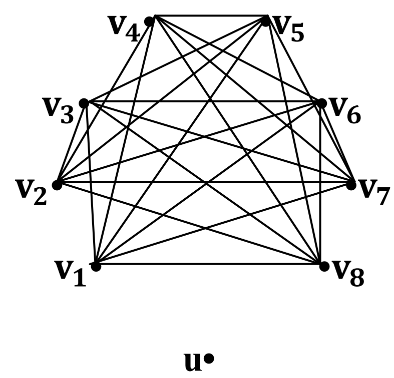
Then, the result follows from (1) and (2). ■

Example



C_3^4

Figure 2.21



$\overline{C_3^4}$

Figure 2.22

For the graph C_3^4 in figure 2.21, the vertex set D = $\{u, v_1, v_2\}$ is a $\gamma_{\overline{cc}}$-set and hence $\gamma_{\overline{cc}}(C_3^4) = 3$.

Theorem: 2.2.11

For any Corona graph G^+, $\gamma_{\overline{cc}}(G^+) = \frac{p}{2}$, where p is the order of G.

Proof:

Since G is the connected graph with order n gives G^+ has 2n vertices. That is, p = 2n.

For the domination of G^+, the minimum dominating set must contain all the n-vertices of G.

Also the same dominating set itself is $\gamma_{\overline{cc}}$-ccd set of G^+, and

Hence $\gamma_{\overline{cc}}(G^+) = n = \frac{p}{2}$. ■

Proposition 2.2.12 For the cycle graph C_n, $n \geq 3$, $\gamma_{\overline{cc}}(C_n) = n - 2$.

Proposition 2.2.13 For the path graph P_n, $n \geq 2$, $\gamma_{\overline{cc}}(P_n) = n - 2$.

Proposition 2.2.14 For the triangle snake graph mC_3, $m \geq 1$,

$$\gamma_{\overline{cc}}(mC_3) = \begin{cases} 3, m \leq 3 \\ m-1, m \geq 4 \end{cases}$$

Proposition 2.2.15 For the lollipop graph $L_{m,n}$, $m \geq 3$, $n \geq 1$,

$\gamma_{\overline{cc}}(L_{m,n}) = n$.

Proposition 2.2.16 For the quadrilateral snake graph mC_4, $m \geq 1$,

$\gamma_{\overline{cc}}(mC_4) = m + 1$.

The following theorem gives bound for $\gamma_{\overline{cc}}(G)$.

Theorem: 2.2.17

If G is a graph with $p \geq 3$ vertices, $\gamma_{\overline{cc}}(G) = p - \varepsilon_T(G)$

Proof:

Let T be a spanning tree of G with $\varepsilon_T(G)$ end vertices.

Let S be the set of end vertices in T.

Then T-S is a connected dominating set.

Thus $\gamma_{\overline{cc}}(G) \leq |T - S| = p - \varepsilon_T(G)$ … (1)

Conversely,

Let D be a $\gamma_{\overline{cc}}$ - set. Since $\langle D \rangle$ is connected, $\langle D \rangle$ has a spanning tree T_1.

A spanning tree T of G is formed by adding the remaining $p - \gamma_{\overline{cc}}(G)$ vertices of V - D to T_1 and adding edges of G such that each vertex in V – D is adjacent to exactly one vertex in D.

Now T has at least $p - \gamma_{\overline{cc}}(G)$ end vertices.

Hence $\gamma_{\overline{cc}}(G) \geq p - \varepsilon_T(G)$ …(2)

From (1) & (2),

$$\gamma_{\overline{cc}}(G) = p - \varepsilon_T(G) \quad \blacksquare$$

Corollary 2.2.18

For any non complete graph G with p ≥ 3 vertices, $\gamma_{\overline{cc}}(G) \leq p - 2$.

Corollary 2.2.19

If T is a tree and p ≥ 3, then $\gamma_{\overline{cc}}(G) = p - e$, where e denote the number of end vertices of a tree T.

Theorem: 2.2.20

For any graph G,

$$p/(\Delta(G) + 1) \leq \gamma_{\overline{cc}}(G) \leq 2q - p$$

Furthermore, the lower bound is attained if and if only $\Delta(G) = p-1$ and the upper bound is attained if and if only G is a path.

Proof:

Since $p/(\Delta(G) + 1) \leq \gamma(G)$ and $\gamma(G) \leq \gamma_c(G)$.

From 3.2.1, $\gamma(G) \leq \gamma_{\overline{cc}}(G)$ the lower bound holds.

It is easy to see that this lower bound is attained if and if only G has a vertex of degree p-1.

We now show that the upper bound holds. Since G is connected, q ≥ p – 1 and Corollary 2.2.18,

$$\gamma_{\overline{cc}}(G) \leq p - 2 = 2(p - 1) - p$$

Hence, $\gamma_{\overline{cc}}(G) \leq 2q - p$

We now show that $\gamma_{\overline{cc}}(G) = 2q - p$ if and only if G is a path.

If G is a path, then from corollary 2.2.19

$\gamma_{\overline{cc}}(G)$ = p – 2 = 2(p -1) – p = 2q – p.

Conversely, suppose $\gamma_{\overline{cc}}(G)$ = 2q – p

Then by corollary 2.2.18,

We have 2q – p ≤ p – 2 which implies q ≤ p – 1.

Since G is connected, G must be a tree with q = p – 1.

Thus corollary 2.2.19, $\gamma_{\overline{cc}}(G)$ = p – e

If e > 2, then $\gamma_{\overline{cc}}(G)$ = p – e < p – 2 = 2q – p, a contradiction which shows that e ≤ 2.

But, since G is a tree, e ≥ 2.

Thus e = 2 and G is a path. ■

Theorem: 2.2.21

For any graph G, $\gamma_{\overline{cc}}(G) \leq 2p - \Delta(G)$

Proof:

Let D be the ccd-set of G. Since $\langle D \rangle$ is connected and

$\gamma_c(G) \leq$ p - $\Delta(G)$.

We have D has at most p - $\Delta(G)$ vertices.

But D is also a dominating set of $\bar{G}$, We have D has atmost p vertices and

Hence $\gamma_{\overline{cc}}(G) \leq |D| \leq p - \Delta(G) + p = 2p - \Delta(G)$

$\Rightarrow$ $\gamma_{\overline{cc}}(G) \leq 2p - \Delta(G)$ ■

Theorem 2.2.22

For any tree T, $\gamma_{\overline{cc}}(T) \leq$ p - $\Delta(T)$ + 1

Proof:

Let D be the ccd-set of T. Since $\langle D\rangle$ is connected and $\gamma_c(T) \leq$ p - $\Delta(T)$.

We have D has at most p - $\Delta(T)$ vertices.

But D is also a dominating set of $\bar{T}$, We have D has atleast one vertex.

Hence $\gamma_{\overline{cc}}(T) \leq |D| \leq$ p - $\Delta(T)$ + 1 = p - $\Delta(T)$ + 1

$\Rightarrow$ $\gamma_{\overline{cc}}(T) \leq$ p - $\Delta(T)$ +1 ■

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