EQUALITY OF EDGE DOMINATION PARAMETERS IN GRAPHS

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Abstract : Let G be a (p,q) –graph with edge domination number γ' and total edge domination number γ_t'. In this paper we investigate the structure of graphs in which some of the edge domination parameters are equal. We characterize graphs for which $\gamma' = \gamma_t'$.

Key words: Edge domination number, Connected edge domination number and Total edge domination number.

1. Introduction

By a graph G = (V,E) we mean a finite undirected graph without loops or multiple edges. Terms not defined here are used in the sense of Harary [1].

The concept of edge domination was introduced by Mitchell and Hedetniemi. A subset X of E is called an edge dominating set of G if every edge not in X is adjacent to some edge in X. The edge domination number $\gamma'(G)$ (or γ' for short) of G is the minimum cardinality taken over all edge dominating sets of G. An edge dominating set X of is called a total edge dominating set of G if the induced subgraph $\langle X \rangle$has no isolated edges. The total edge domination number $\gamma_t'(G)$($or \gamma_t'$ for short) of G is the minimum cardinality taken over all total edge dominating sets of G.

Allan and Laskar [2] proved that for any $K_{1,3} -$ free graph, the domination number and independent domination number are equal. Topp and Volkmann [3] generalized the result of Allan and Laskar and constructed new classes of graphs with equal domination and independent domination number.

Harary and Livingston [4] characterized caterpillars with equal domination and independent domination number. In [5] they gave the characterization of trees with equal domination and independent domination number.

Payan and Xuong [6] proved that for any graph G on 9 vertices, $\gamma = \overline{\gamma = 3}$ if and only if $G = K_3 \times K_3$. Arumugam and

Paulraj Joseph [7] studied the class of graphs for which connected domination number and domination number are equal.

In this paper we initiate a study of graphs in which some of the edge domination parameters are equal. We characterize graphs for which $\gamma' = \gamma'_t$.For this we need the following Theorem.

Theorem 1.1 [8]Let *P* denote the property that $\gamma' = \gamma'_c = n.\ \gamma' = \gamma'_t = n.$ A connected graph *G* is *P – critical* if and only if *G* is isomorphic to $S(K_{1,n})$

2. Main Results

Lemma 2.1 Let *G* be a graph without isolated edges. If $\gamma' = \gamma'_{t'}$ then for any minimum total edge dominating set *S* of *G*, each component of $\langle S \rangle$ is isomorphic to a star.

Proof Suppose $\gamma' = \gamma'_{t'}$. Let *S* be any minimum total edge dominating set of *G*. If a component of $\langle S \rangle$ contains two vertices u, v of degree ≥ 2, then $S \setminus \{e\}$ where *e* is any non-pendent edge of $\langle S \rangle$ incident with u forms an edge dominating set of *G* so that $\gamma' < \gamma'_{t'}$ which is a contradiction. Hence each component of $\langle S \rangle$ is isomorphic to a star.

Definition 2.2 Let *P* denote the property that $\gamma' = \gamma'_t = n.$ A connected graph *G* is said to be *P – critical* if *G* satisfies *P* and no proper subgraph H of *G* without isolated edges satisfies *P*.

Lemma 2.3 Let *P* denote the property that $\gamma' = \gamma'_t = n.$ A connected graph *G* is *P – critical* if and only if *G* is isomorphic to $S(K_{1,n})$ or the graph G_1 obtained from $S(K_{1,n})$ by identifying two of its pendent vertices.

ProofLet *G* be a connected graph which is *P – critical* so that $\gamma' = \gamma'_t = n.$ Let *S* be any minimum total edge dominating set of *G*. We claim that $\langle S \rangle$ is connected.

Suppose $\langle S \rangle$ is not connected. Let $S_1, S_2, \ldots, S_k$ be the components of $\langle S \rangle$. ByLemma 2.1, each S_i is isomorphic to K_{1,n_i}. Let $V(S_i) = \{u_i, u_{i1}, u_{i2}, \ldots, u_{in_i}\}, \deg_{S_i} u_i = n_i$ and $n_1 + n_2 + \ldots + n_k = n$. Let $e_{ij} = u_i u_{ij}$, $j = 1, 2, \ldots, n$. We claim that each e_{ij} in S, there exists exactly one edge x_{ij} in *G* such that x_{ij} is adjacent to e_{ij} but not adjacent to any other edge of *S*. Since $|S| = \gamma' = \gamma'_t$, S is a minimum edge dominating set of *G* and hence such an x_{ij} exists.

Suppose there exist two edges $x_{ij} = u_{ij}w_1$ and $y_{ij} = u_{ij}w_2$ in *G* such that x_{ij} and y_{ij} are adjacent to e_{ij} but not adjacent to any other edges of *S*. If w_1 or w_2, say w_1, is a pendent vertex of *G*, then $G - w_1$ has no isolated edges and $\gamma'(G - w_1) = \gamma'_t(G - w_1) = n$ which is a contradiction. Hence both w_1 and w_2 have degree at least 2. Since *S* is a minimum total edge dominating set of *G*, w_1 is adjacent to a vertex in S_m(*m* may be equal to *i*). Now $G - x_{ij}$is a proper subgraph of *G* without isolated edges and $\gamma'(G - x_{ij}) = \gamma'_t(G - x_{ij}) = n$ which is a contradiction. Hence for each e_{ij} in *S*, there exists exactly one edge x_{ij}in *G* such that x_{ij} is adjacent to e_{ij} but not adjacent to any other edges of *S*.

Let H_i be the subgraph of G induced by the edges $e_{i1}, e_{i2}, ..., e_{in_i}, x_{i1}, x_{i2}, ..., x_{in_i}$. Then $\gamma'(H_i) = \gamma'_t(H_i) = n_i$. By Theorem 1,1, H_i is isomorphic to $S(K_{1,n_i})$ or the graph G_i obtained from $S(K_{1,n_i})$ by identifying two of its pendent vertices. Let $H = H_1 \cup H_2 \cup .... \cup H_k$.

Suppose that the edges x_{mi} and x_{nj} $(i \neq j)$ are adjacent in H. Since x_{nj} isthe only edge of G adjacent to e_{nj} but not adjacent to any other edges of S, $(S \setminus \{e_{mi}, e_{nj}\}) \cup \{x_{mi}\}$ is an edge dominating set of G of cardinality n – 1 which is a contradiction. Hence each component H_i of H isomorphic to $S(K_{1,n_i})$ or the graph G_i obtained from $S(K_{1,n_i})$ by identifying two of its pendent vertices. Now H is a proper subgraph of G and $\gamma'(H) = \gamma'_t(H) = n$ which is a contradiction. Hence $\langle S \rangle$ is connected so that $\gamma' = \gamma'_t = \gamma'_c = n$ and the result follows from Theorem 1.1

The converse is obvious.

Theorem 2.4 Let P denote the property that $\gamma' = \gamma'_t =$ n. Let G be a graph without isolated edges. Then $\gamma' = \gamma'_t =$ n if and only if G satisfies the following conditions.

(i) G contains a subgraph H with components $H_1, H_2, ..., H_k$ such that each H_i is isomorphic to $S(K_{1,n_i})$ or the graph G_{n_i} obtained by identifying two of its pendent vertices of $S(K_{1,n_i})$ where $n_i \geq 2$ and $n_1 + n_2 + ... + n_k = n$.

(ii) Every edge e in $E(G) \setminus E(H)$ has at least one end in $V(H_1)$ where $H_1 = \cup K_{1,n_i}$ is the subgraph of H induced by a minimum total edge dominating set of H.

(iii) If there exists an edge e of $E(G) \setminus E(H)$ joining two pendent vertices of K_{1,n_i} in H_i, then the centre of K_{1,n_i} has degree at least three in G.

(iv) If there exists an edge e = uv in $E(G) \setminus E(H)$ such that u is a pendent vertex of K_{1,n_i} for some i, then either v is a vertex of $V(G) \setminus V(H)$ or v is a center of K_{1,n_j} in H_j for some $j \neq i$ or v is a vertex of $V(H_j) \setminus V(K_{1,n_j})$ for some j (j may be equal to i) and in this case the pendent vertices of K_{1,n_j} adjacent to v have at least 3 in G.

Proof: Suppose $\gamma' = \gamma'_t =$ n. Let S be any minimum total edge dominating set of G. By lemma 2.1, $H_1 = \langle S \rangle = \bigcup_{i=1}^{k} K_{1,n_i}$ where $n_i \geq 2$ and $n_1 + n_2 + ... + n_k = n$.

Let $V(K_{1,n_i}) = \{u_i, u_{i1}, u_{i2}, ..., u_{in_i}\}$ and $\deg_{H_i} u_i = n_i$. Let $e_{im} = u_i u_{im}$. Now as in Lemma 2.3, for each edge e_{im} in S, we can choose an edge x_{im} is adjacent to e_{im} but not adjacent to any other edge of S and the subgraph H induced by e_{im}'s and x_{im}'s is P – *critical*. Hence (i) holds.

Also since S is a minimum total edge dominating set of G, (ii)holds.

Suppose there exists an edge e in $E(G) \setminus E(H)$ joining two pendent vertices, say u_{i1}, u_{i2} of K_{1,n_j} and $\deg_G u_i = 2$. Then $(S \setminus \{e_{i1}, e_{i2}\}) \cup \{u_{i1}, u_{i2}\}$ is an edge dominating set of G of cardinality n–1, which is a contradiction. Thus G satisfies (iii).

Now let e = uv be an edge in E(G) \ E(H) such that u is a pendent vertex of K_{1,n_i} for some i. If v is a pendent vertex of K_{1,n_j} $(i \neq j)$, then $(S \setminus \{u_i u, u_j v\}) \cup \{uv\}$ is an edge dominating set of G, which is a contradiction. If v is a vertex of V(H_j) \ V(K_{1,n_j}) (j may be equal to i) and a pendent vertex w of K_{1,n_j} adjacent to v has degree 2 in G, then $(S \setminus \{u_i u, u_j v\}) \cup \{uv\}$ is an edge dominating set of G, which is a contradiction. Thus G satisfies (iv).

Conversely suppose that G satisfies (i), (ii), (iii) and (iv). Then E(H_1) is a minimum total edge dominating set of G and also a minimum edge dominating set of G so that $\gamma' = \gamma'_t =$ n.

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