ON TERNARY QUADRATIC DIOPHANTINE EQUATION $3(x^2+y^2)-4xy=42z^2$

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ABSTRACT

The ternary homogeneous quadratic equation given by $3(x^2+y^2)-4xy=42z^2$ representing a cone is analyzed for its non-zero distinct integer solutions. A few interesting relations between the solutions and special polygonal and pyramidal numbers are presented.

KEY WORDS
Ternary quadratic integer solutions, figurate numbers, Special numbers.
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INTRODUCTION

The Diophantine equations offer an unlimited field for due to their variety[1-3]. In particular, one may refer [4-15**]** for quadratic equations with three unknowns. This communication concerns with yet another interesting homogenous quadratic equation with three unknowns $3(x^2+y^2)-4xy=42z^2$ for determining its infinitely many non –zero integral points. Also, a few interesting relations among the solutions are presented.

Notations:

$\Pr_A$ -Pronic number of rank A

OH_A -Octahedral number of rank A

$t_{m,n}$ -Polygonal number of rank n with size m

$CP_{m,n}$ -Centered Pyramidal number of rank n with size m

METHOD OF ANALYSIS

The Diophantine equation representing the ternary quadratic equation to be solved for its non-zero distinct integral solutions is

$$3(x^2+y^2)-4xy=42z^2 \quad (1)$$

Note that (1) is satisfied by the following non-zero integer triples

(-26,-6,6), (-22,-30,6), (-6,18,6),(480,352,96),(-198,-186,42),(48,80,16),(48,-144,48), (-78,- 18,18), (402,334,82),

However, we have other solutions for (1), which are illustrated below.

The substitution of linear transformations

$$\left.\begin{aligned} x &= u+v \\ y &= u-v \end{aligned}\right\} ,\ u \neq v \neq 0 \tag{2}$$

in (1) gives

$$u^2 + 5v^2 = 21z^2 \tag{3}$$

Assume that $z = z(a,b) = a^2 + 5b^2$, $a,b \neq 0$ (4)

We illustrate below methods of finding different patterns of integer solutions to (1).

PATTERN 1:

Note that

$$21 = \left(1+i2\sqrt{5}\right)\left(1-i2\sqrt{5}\right) \tag{5}$$

Substituting (4) and (5) in (3) and applying the method of factorization, define

$$u + i\sqrt{5}v = \left(1+i2\sqrt{5}\right)\left(a+i\sqrt{5}b\right)^2$$

Equating the real and imaginary parts, we get

$$\begin{aligned} u &= a^2 - 5b^2 - 20ab \\ v &= 2a^2 - 10b^2 + 2ab \end{aligned} \tag{6}$$

Substituting (6) in (2), we get

$$\begin{aligned} x &= x(a,b) = 7a^2 - 77b^2 - 10ab \\ y &= y(a,b) = 5a^2 - 55b^2 - 34ab \end{aligned} \tag{7}$$

Thus (4) and (7) represent non-zero distinct integral solutions to (1) in two parameters.

PROPERTIES:

1. $x(A, A+1) + 3y(A, A+1) = -84\,\mathrm{Pr}_A$
2. $x(A, 2A^2+1) + 3y(A, 2A^2+1) = -252\,OH_A$
3. $x(A, A) + 3y(A, A) = -84t_{4,A}$

Remark:

In addition to (5), 21 may also be represented by $21 = \dfrac{\left(32+i\sqrt{5}\right)\left(32-i\sqrt{5}\right)}{49}$

Proceeding as in pattern-1, the corresponding integer solutions to (1) are

$$x = 231a^2 - 1155b^2 - 378ab$$

$$y = 217a^2 - 1085b^2 - 518ab$$

$$z = 49a^2 + 245b^2$$

PATTERN 2:

(3) can be expressed in the form of ratio as

$$\frac{u+4z}{5(z+v)} = \frac{(z-v)}{u-4z} = \frac{A}{B},\ B \neq 0$$

which is equivalent to the following two equations.

$$(4B-5A)z + Bu - 5Av = 0$$
$$(B+4A)z - Au - vB = 0$$

Applying the method of cross multiplication and using (2), the corresponding integer solutions to (1) are given by

$$x = x(A,B) = -15A^2 + 3B^2 - 18AB$$
$$y = y(A,B) = -25A^2 + 5B^2 - 2AB$$
$$z = z(A,B) = A^2 + 5B^2$$

PROPERTIES:

1. $x(A,1) + 15\,\Pr_A \equiv 0 \pmod 3$
2. $2[x(A,1) + 3z(A,1) + 18\,\Pr_A]$, a nasty number
3. $z(A,A) - y(A,A)$, a perfect square
4. $y\{(A,A+1),2A\} + 25t_{4,A}^2 + 54CP_{6,A} + 9t_{4,A} = 0$

PATTERN 3:

Introducing the linear transformations

$$u = 4U$$
$$\left.\begin{array}{l} z = X + 5T \\ v = X + 21T \end{array}\right\} \qquad (8)$$

in (3), we have

$$X^2 = U^2 + 105T^2 \qquad (*)$$

which is satisfied by

$$\begin{array}{l} T = 2AB \\ U = 105A^2 - B^2 \\ X = 105A^2 + B^2 \end{array} \qquad (9)$$

Substituting (9) in (8) and employing (2), the integer solutions of (1) are given by

$$x = x(A,B) = 525A^2 - 3B^2 + 42AB$$
$$y = y(A,B) = 315A^2 - 5B^2 - 42AB$$
$$z = z(A,B) = 105A^2 + B^2 + 10AB$$

Note:

Instead of (8), one may also consider the transformations as $\begin{aligned} z &= X - 5T \\ v &= X - 21T \end{aligned}$ (8a)

For this choice, the corresponding integer solutions are

$$x = x(A,B) = 525A^2 - 3B^2 - 42AB$$
$$y = y(A,B) = 315A^2 - 5B^2 + 42AB$$
$$z = z(A,B) = 105A^2 + B^2 - 10AB$$

PATTERN 4:

(*) can be written as

$$X + U = T^2$$
$$X - U = 105$$

And solving, we get

$$X = 2A^2 + 2A + 53$$
$$T = 2A + 1$$
$$U = 2A^2 + 2A - 52$$

Substituting the above values of X,T,U in (8), (8a) and employing (2), the corresponding two sets of integer solutions to (1) are as follows

SET I:
$$x = x(A) = 10A^2 + 52A - 134$$
$$y = y(A) = 6A^2 - 36A - 282$$
$$z = 2A^2 + 12A + 58$$

SET II:
$$x = x(A) = 10A^2 + 32A - 176$$
$$y = y(A) = 6A^2 + 48A - 240$$
$$z = 2A^2 - 8A + 48$$

PATTERN 5:

It is worth to note that (*) can also be written as the system of double equations as follows:

$SystemI$	$SystemII$	$SystemI$
$X + U = 21$	$X + U = 21T^2$	$X + U = 21T$
$X - U = 5T^2$	$X - U = 5$	$X - U = 5T$

Solving each of the above systems, the values of X,T,U are obtained. Using these values respectively in (8), (8a) and employing (2), one obtains the corresponding integer solutions to (1).

PATTERN 6:

Equation (3) can be written as

$$u^2 + 5v^2 = 21z^2 * 1 \quad (10)$$

Note that $21 = (4 + i\sqrt{5})(4 - i\sqrt{5}); 1 = \frac{(2 + i\sqrt{5})(2 - i\sqrt{5})}{9}$ (11)

Substituting (4), (11) in (10) and using the method of factorization, define

$$u + i\sqrt{5}v = (4 + i\sqrt{5})(a + i\sqrt{5}b)^2 \left(\frac{2 + i\sqrt{5}}{3} \right)$$

Equating the real and imaginary parts, we get

$$u = a^2 - 5b^2 - 20ab$$

$$v = 2a^2 - 10b^2 + 2ab$$

In view if (2), note that

$$x = 3a^2 - 15b^2 - 18ab$$

$$y = -a^2 + 5b^2 - 22ab \quad (12)$$

$$z = a^2 + 5b^2$$

Thus (12) represents non-zero distinct integer solutions to (1)

Remark: Rewrite (11) as

$$21 = \frac{(32 + i\sqrt{5})(32 - i\sqrt{5})}{49}; 1 = \frac{(2 + i\sqrt{5})(2 - i\sqrt{5})}{9}$$

Proceeding as in pattern-6, the corresponding integer solutions to (1) are

$$x = 1953a^2 - 9765b^2 - 4662ab$$

$$y = 525a^2 - 2625b^2 - 9618ab$$

$$z = 441a^2 + 2205b^2$$

CONCLUSION:

In this paper, we have obtained infinitely many non-zero distinct integer solutions to the ternary quadratic Diophantine equation represented by $3(x^2 + y^2) - 4xy = 42z^2$**.** As quadratic equations are rich in variety, one may search for other choices of quadratic equations with variables greater than or equal to 3 and determine their properties through special numbers.

REFERENCES

1. Dickson LE.History of Theory of Numbers and Diophantine Analysis,Vol 2,Dove publications,New York 2005.
2. Mordell LJ. Diophantine Equations” Academic Press, Newyork 1970.
3. Carmicheal RD.The Theory of Numbers and Diophantine Analysis , Dover publications,,Newyork 1959.
4. Gopalan MA,Geetha D. Lattice points on the Hyperboloid of two sheets $x^2 - 6xy + y^2 + 6x - 2y + 5 = z^2 + 4$ Impact J Sci Tech 2010;4:23-32.

5. Gopalan MA ,Vidhyalakshmi S,Kavitha A,Integralpoints on the Homogeneous cone $z^2 = 2x^2 - 7y^2$,The Diophantus J Math 2012;1(2):127-136.
6. Gopalan MA,Vidhyalakshmi S, Sumathi G.Lattice points on the Hyperboloid of one sheet $4z^2 = 2x^2 + 3y^2 - 4$. Diophantus J Math 2012; 1(2): 109-115.
7. Gopalan MA, Vidhyalakshmi S,Lakshmi K. Integral points on the Hyperboloid of two sheets $3y^2 = 7x^2 - z^2 + 21$. Diophantus J Math 2012; 1(2): 99-107.
8. Gopalan MA,Vidhyalakshmi S, Mallika S.Observations on Hyperboloid of one sheet $x^2 + 2y^2 - z^2 = 2$. Bessel J Math 2012; 2(3): 221-226.
9. Gopalan MA ,Vidhyalakshmi S,Usha Rani TR , Mallika S,Integral points on the Homogeneous cone $6z^2 + 3y^2 - 2x^2 = 0$,The Impact J Sci Tech 2012;6(1):7-13.
10. Gopalan MA ,Vidhyalakshmi S,Usha Rani TR,Integral points on the non-homogeneous cone $2z^2 + 4xy + 8x - 4z = 0$, Global Journal of Mathamatics and Mathamatical sciences 2012;2(1):61- 67 .
11. Gopalan MA ,Vidhyalakshmi S,Uma Rani J, Integral points on the Homogeneous cone $4y^2 + x^2 = 37z^2$,Cayley J of Math 2013;2(2):101- 107.
12. Gopalan MA,Vidhyalakshmi S, Kavitha A. Observations on the Hyperboloid of two sheet $7x^2 - 3y^2 = z^2 + z(y - x) + 4$. International Journal of Latest Research in Science and technology 2013; 2(2): 84-86.
13. Gopalan MA ,Sivagami B. Integral points on the homogeneous cone $z^2 = 3x^2 + 6y^2$. IOSR Journal of Mathematics 2013; 8(4): 24-29.
14. Gopalan MA,Geetha V. Lattice points on the homogeneous cone $z^2 = 2x^2 + 8y^2 - 6xy$. Indian journal of Science 2013; 2: 93-96.
15. Gopalan MA, Vidhyalakshmi, Santhi J. Lattice points on the homogeneous cone $8(x^2 + y^2) - 15xy = 56z^2$. Sch Journal of Phy Math Stat 2014; 1(1): 29-32.