COTOTAL BLOCK SUBDIVISION DOMINATION IN GRAPHS

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Abstract:

For any graph $G(V,E)$,block subdivision graph $B[S(G)]$ is a graph whose set of vertices is the union of the set of blocks of $S(G)$ in which two vertices are adjacent if and only if the corresponding blocks of $S(G)$ are adjacent . A dominating set D of a graph $B[S(G)]$ is a cototal block dominating set, if the induced subgraph $\langle V[BS(G)] - D\rangle$ has no isolated vertices .The cototal block subdivision domination number $\gamma_{cbs}(G)$ is the minimum cardinality of a cototal block subdivision dominating set of G. In this paper many bonds on $\gamma_{cbs}(G)$ are obtained in terms of elements of G but not the elements of $B[S(G)]$. Also its relation with cubic graph and other domination parameters were established.

1. Introduction:

In this paper, we follow the notation of F.Harary[2]. All the graphs considered here are finite and simple. As usual p, q,and n denotes the number of vertices, edges and blocks of a graph G respectively.The minimum degree of a vertex in G is denoted by $\delta(G)$. A graph G is a tree (T), if it is connected having no cycles. A vertex v is called a cut vertex, if removing it from G increases the number of components of G. For any real x,x denotes the smallest integer not less than x. A graph G is called a trivial, if it has one vertex and no edges. If G has atleast one edge, then G is called non trivial graph. A non trivial connected graph G with atleast one cut vertex is called a separable graph, otherwise a non separable graph. The vertex independence number $\beta_0(G)$ of a graph G is the maximum cardinality of an independent set of vertices in G. Every cubic graph G having degree three with even number of vertices.

We begin with some basic definitions in domination theory.

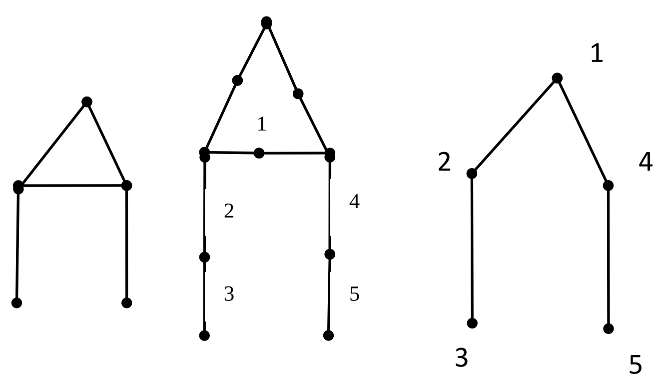
A set of vertices $D \subseteq V(G)$ of a graph G is a dominating set, if every vertex in $V - D$ is adjacent to some vertex in D.The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set in G.This concept was introduced by T.W.Haynes[3]. A dominating set D is a total dominating set, if the induced subgraph $\langle D \rangle$ has no isolated vertices. The total domination number $\gamma_t(G)$ of a graph G is the minimum cardinality of a total dominating set. This concept was introduced by cockayne, Dawes and Hedetniemi [1].

A dominating set D is a connected dominating set, if the induced subgraph $\langle D \rangle$ is conncted. The connected domination number $\gamma_C(G)$ of a connected graph G is the minimum cardinality of a connected dominating set. This concept was introduced by sampath kumar and Walikar [7]. A dominating set D of G is a cototal dominating set if the induced subgraph $\langle V - D \rangle$ has no isolated vertices. The cototal domination number $\gamma_{cot}(G)$ of G is the minimum cardinality of a cototal dominating set. This concept was introduced by Kulli, Janakiram and Iyer [4] and [6].

A dominating set D of $B[S(G)]$ is a cototal dominating set, if the induced subgraph $\langle V[BS(G)] - D \rangle$ has no isolated vertices .The cototal block subdivision domination $\gamma_{cbs}(G)$ of $B[S(G)]$ is the minimum cardinality of a cototal dominating set.

A Subdivision of an edge uv is obtained by removing edge uv, adding a new vertex w and adding edges uw and wv. The subdivision graph $S(G)$ of a graph G is the graph obtained from G by subdividing each edge of G.

The following figures illustrate the formation of a block subdivision graph $B[S(G)]$ of a graph G.



G S(G) B[S (G)]

In this paper, some results on $\gamma_{cbs}(G)$ were obtained interms of vertices,blocks,cubic graphs and other parameters of G.Also we obtain some results on $\gamma_{cbs}(G)$ with other domination parameter of G.

We need the following Theorems for our further results.

2) Main Results:

Theorem A [5] :A dominating set D is a minimal dominating set if and only if for every vertex v in D, one of the following condition holds.

i) v is an isolated vertex of D.

ii) There exists a vertex u in $V - D$ such that $N(u) \cap D = \{v\}$.

Theorem B[4]: A cototal dominating set D of G is minimal if and only if for vertex $v \in D$,one of the following holds.

i) There exists a vertex $u \in V - D$such that $N(u) \cap D = \{v\}$.

ii) v is an isolated vertex in $\langle D \rangle$.

iii) v is an isolated vertex in $\langle (V - D) \cup \{v\} \rangle$.

Now we consider the upper bound on $\gamma_{cbs}(G)$ in terms of blocks in G.

Theorem 2.1: For any graph G with n –blocks, then $\gamma_{cbs}(G) \leq n + 1$.

Proof: Suppose $B[S(G)] = M$ be a block subdivision graph of a graph G. Let $H = \{B_1, B_2, B_3, \dots \dots \dots B_n\}$ be the blocks of G and $H_1 = \{B_1^1, B_2^1, B_3^1, \dots \dots \dots B_n^1\}$ be the blocks of subdivision of G. Let$B = \{v_1, v_2, v_3, \dots \dots \dots v_n\}$ be the set of vertices of M which corresponds to the blocks of H_1.Now we consider the following cases.

Case(1): Suppose every vertex of G lies on atmost two blocks of G.Then M is a tree. Let $B_1 = \{v_1, v_2, v_3, \dots \dots \dots v_s\}$ be the set of all end vertices of M and $B_1^1 = V[M] - B_1$. If $N(B_1) \cap N(B_1^1) = \emptyset$ then $B_1 \cup B_1^1$ forms a dominating set. If $\langle V[M] - [B_1 \cup B_1^1 \rangle$ has no isolates, then $\{B_1 \cup B_1^1\}$ is a $\gamma_{cbs} - set$ of M.Thus $|B_1 \cup B_1^1| \leq n + 1$ gives $\gamma_{cbs}(G) \leq n + 1$.

Case (2): Suppose every cut vertex of G lies on atleast three blocks. Then M is not a tree. Let$S_1 = \{v_1, v_2, v_3, \dots \dots \dots v_m\}$ be the set of all end vertices of M and $S_2 = \{v_1, v_2, v_3, \dots \dots \dots v_l\}$ be the set of vertices with $\deg(v_i) \geq 2, \forall v_i \in S_2, 1 \leq i \leq l$. Suppose $S_2^1 \subseteq S_2, \forall v_j \in S_2^1$ and $N[S_2^1] = V[B - S_1]$. Then S_2^1 is a dominating set of $\{B - S_1\}$.Hence $\{S_1\} \cup \{S_2^1\}$ forms a dominating set of M in which $\langle M - \{S_1\} \cup \{S_2^1\} \rangle$ has no isolates. Clearly $|\{S_1\} \cup \{S_2^1\}| \leq n + 1$ gives $\gamma_{cbs}(G) \leq n + 1$.

In the following theorem, we obtain a sharp lower bound for $\gamma_{cbs}(G)$ in terms of cototal domination in G.

Theorem 2. 2: Let G be a tree, then $\gamma_{cbs}(G) + 1 \geq \gamma_{cot}(G)$. Equality holds for G is a star.

Proof: Suppose $B[S(G)] = M$ be a block subdivision of a graph G.Then we consider a set S^1 with $\deg(v_i) \geq 2$, $\forall\, v_i \in S^1$ in M.

Let$V = \{v_1, v_2, v_3, \dots \dots \dots v_n\}$ be the set of vertices in G.Suppose$S = \{v_1, v_2, v_3, \dots \dots \dots v_i\} \subseteq V(G)$ be the minimal set of vertices which covers all the vertices in G such that $\langle V - S \rangle$ has no isolates. Then $|S| = \gamma_{cot}(G)$.

Consider a set $S_1^1 = \{v_1, v_2, v_3, \ldots \ldots \ldots v_j\} \subseteq S^1$ with the property that $d(u,v) \geq 2, \forall\, u, v \in S_1^1$ which covers all the vertices of $V[M] - S_2, \forall\, v_k \in S_2$ has deg $(v_k) = 1$. Clearly $\{S_1^1\} \cup \{S_2\}$ forms a $\gamma - set$ of M.Suppose the subgraph$\langle V[M] - \{S_1^1\} \cup \{S_2\}\rangle$ has no isolates. Then $\{S_1^1\} \cup \{S_2\}$ forms a minimal cototal dominating set of M.Otherwise we consider the following cases.

Case(1): If M contains end vertices. Then in this case $S_2 = \{v_1, v_2, v_3, \ldots \ldots \ldots v_m\}$ be the set of all end vertices. Further more, Let $S^{11} \subseteq S^1$ be the set of vertices with distance $(x,y) \geq 2, \forall\, x \in S_2, y \in S^{11}$ such that $S_2 \cup S^{11} = D$ covers all the vertices in M and the subgraph $\langle V(M) - D\rangle$ does not contain any isolated vertex. Clearly D is a $\gamma_{cbs} - set$ of M.

Case (2): If M does not contain any end vertex. Then in this case, Let $S_2^{11} \subseteq S^1$ and $S_2^{11} \in N(S_2^1)$. Clearly $S_2^1 \cup S_2^{11} = D$, $S_2^{11} \subseteq S^1$ covers all the vertices in M such that the induced subgraph $\langle V(M) - D\rangle$ has no isolated vertex. Hence D itself is a $\gamma_{cbs} - set$ of M. Since $E(G) = V(M), D$ forms a minimal cototal dominating set of M.Therefore, it follows that $|D| + 1 \geq |S|$ gives $\gamma_{cbs}(G) + 1 \geq \gamma_{cot}(G)$.

Suppose $G = K_{1,n}$.Then in this case $|D| = n$, $|S| = n + 1$.Since for any tree $\delta(G) = 1$, it follows that $|D| + 1 \geq |S|$ gives the above result.

The following result is a relation between $\gamma_{cbs}(T)$, domination and vertices of T.

Theorem 2.3: For any connected $(p, q\,)$ tree $T, \gamma_{cbs}(T) + \gamma(T) \geq P$.

Proof: let $V = \{v_1, v_2, v_3, \ldots \ldots \ldots v_n\}$ be the set of vertices in G.Suppose $V_1 \subseteq V(G)$ be the set of vertices with $\deg(v) \geq 2$, $\forall\, v \in V_1$.Then there exists a minimal vertex set $V_2 \subseteq V_1$ which covers all the vertices in G.Clearly $|V_2| = \gamma(G)$.

Now assume that $S = \{B_1, B_2, B_3, \ldots \ldots \ldots B_n\}$ be the block of G.In subdivision of G, Let $S_1 = \{B_1^1, B_2^1, B_3^1, \ldots\ldots\ldots B_{2n}^1\}$ be the blocks and $S_1^1 = \{u_1^1, u_2^1, u_3^1, \ldots \ldots \ldots .. u_{2n}^1\}$ be the corresponding block vertices in block graph of $S(G)$.

Suppose $S_2^1 = \{u_1^1, u_2^1, u_3^1, \ldots \ldots \ldots .. u_k^1\}$ be the set of all end vertices such that $S_2^1 \subseteq S_1^1$.Let $S_3^1 \subseteq (S_1^1 - S_2^1)\ \forall\, v \in S_3^1$ covers all the vertices of $(S_1^1 - S_2^1)$.Then $D = S_2^1 \cup S_3^1$ with the property $\langle S_1^1 - D\rangle$ has no isolates. Clearly D forms a cototal dominating set.Clearly $|V_2| \cup |D| \geq P$ and hence $\gamma_{cbs}(T) + \gamma(T) \geq P$.

Suppose v be an end vertex in $[BS(G)]$, then v lies in domination of cototal block subdivision of G.

Theorem2. 4: Let v be an end vertex of $B[S(G)]$, then v is in every $\gamma_{cbs} - set$.

Proof : Let $D = \{v_1, v_2, v_3, \ldots \ldots \ldots v_n\} \subseteq V[BS(G)]$ be the minimal cototal dominating set of $B[S(G)]$.Suppose there exists a vertex set $D^{-1} \subseteq V[BS(G)] - D$ be the $\gamma_{cot} - set$ of $B[S(G)]$,assume there exists an end vertex $v \in V[BS(G)]$ .Then we consider any two vertices u and w such that $u, w \notin D^{-1}$. Since $v \notin D^{-1}$,v is in every $u - w$ path in $B[S(G)]$.Further, since $\deg(v) = 1$ where $v \in V[BS(G)]$, it follows that the set $D^1 = (D^{-1} - \{u, w\}) \cup \{v\}$ is also a minimal cototal dominating set of $\gamma_{cot} - set$. Clearly $|D^1| = |D^{-1}| = 1$, a contradiction to the fact that D^{-1} is also a $\gamma_{cot} - set$ of $B[S(G)]$. Hence $v \in D^{-1}$ and v is in every $\gamma_{cbs} - set$.

In the following theorem, we expressed the lower bound for $\gamma_{cbs}(G)$ in terms of cut vertices of G.

Theorem 2.5: For any graph G, then $\gamma_{cbs}(G) + 1 \geq S$.Where S is the number of cut vertices in G.

Proof: Let $v_i \subseteq V(G)$, each vertex of deg $(v_i) \geq 2$ is a cut vertex in G , then $|v_i| = S$.Now $v_s \subseteq V[BS(G)]$ are the end vertices in $B[S(G)]$ and $v_m^1 \subseteq v_m$ is a set of cut vertices in $B[S(G)]$ such that $v_m \subseteq V[BS(G)]$. Suppose the set $\{v_s \cup v_m^1\}$ has no isolates. Then $\langle V[BS(G)] - \{v_s \cup v_m^1\}\rangle$ forms a minimal cototal dominating set in $B[S(G)]$ which gives $|v_s \cup v_m^1| = \gamma_{cbs}(G)$. Otherwise there exists a vertex v in $V[BS(G)] - \{v_s \cup v_m^1\}$ such that by adding to the set $\{v_s \cup v_m^1\} \cup \{v\}$ gives $|\{v_s \cup v_m^1\} \cup \{v\}| = \gamma_{cbs}(G)$. Hence $\{v_s \cup v_m^1\} \cup \{v\} + 1 \geq |v_i|$ gives $\gamma_{cbs}(G) + 1 \geq S$.

The following result is a relationship between$\gamma_{cbs}(T)$, total domination and vertices of G.

Theorem 2. 6: For any connected (p, q) tree$T$, $\gamma_{cbs}(T) \geq P - \gamma_t(T) + 1$.

Proof: Let $\{B_n\}$ be the number blocks of tree T.Then$P = B_n + 1$. Let V be the set of vertices in T and $V_1 \subseteq V$ which are non end vertices in T.Again consider a subset $V_2 \subseteq V_1$, if $V_1 - V_2 = D$ has no isolated vertices. Then D forms a total dominating set, which gives $\gamma_t(T) = |D|$.

Now without loss of generality in $B[S(G)]$.Let $D_1 = \{u_1, u_2, u_3, \ldots\ldots\ldots. u_k\} \subseteq V[BS(G)]$ be the minimum set of vertices such that for every $x \in N(u)$ where $x \in V[BS(G)] - D_1$,$u \in D_1$ and $N(x) \neq \emptyset$ in $V[BS(G)] - D_1$.Clearly D_1 forms a minimal cototal dominating set of $B[S(G)]$.Thus it follows that $|D_1| \geq P - |D| + 1$ and hence $\gamma_{cbs}(T) \geq P - \gamma_t(T) + 1$.

A relationship between the cototal domination in$[BS(G)]$, total domination and independence number of a graph G is established in the following theorem.

Theorem 2.7: For any (p, q) graph G,$\gamma_{cbs}(G) \leq \gamma_t(G) + \beta_O(G)$ where $\beta_o(G)$ is a maximum independence number of G.Equality holds if $G = P_4$.

Proof: Let $K = \{u_1, u_2, u_3, \ldots\ldots\ldots. u_m\} \subseteq V(G)$ be the maximum set of vertices such that $dist(u, v) \geq 2$ and $N(u) \cap N(v) = x$, $\forall\, u, v, \in k$ and $x \in V(G) - K$. Clearly $|K| = \beta_O(G)$.

Let $S = \{u_1, u_2, u_3, \ldots\ldots\ldots. u_l\} \subseteq V(G) - K$ be the minimal set of vertices which cover all the vertices in G.Suppose the subgraph $\langle S\rangle$ has no isolated vertices ,then S itself forms a γ_t-set of G. Otherwise, there exists atleast one vertex $w \in N(s)$ such that $S \cup \{W\}$ forms a minimal total dominating set of G. Let $B = \{b_1, b_2, b_3, \ldots\ldots\ldots. b_n\}$ be the number of blocks of subdivision of G which corresponds to the vertices in $B[S(G)]$ such that $V[BS(G)] = \{u_1, u_2, u_3, \ldots\ldots\ldots. u_n\}$. Now consider

$M = \{v_1, v_2, v_3, \ldots\ldots v_j\} \subseteq V[BS(G)]$. Let F be the set of vertices with $\deg(v) = 1$, $\forall\, v \in F$.Suppose $I = \{v_1, v_2, v_3, \ldots\ldots\ldots. v_m\} \subseteq M$ be the set of vertices such that $diam\,(a, b) \geq 2$ where $a \in F, b \in I$.

Further,$D = F \cup I$ covers all the vertices in $B[S(G)]$. Then D forms a γ_{cot}-set of $B[S(G)]$.Otherwise ,there exists a vertex $z \in N(F) \cup N(I)$ such that $D = F \cup I \cup \{z\}$ forms a minimal cototal dominating set of $B[S(G)]$. Clearly ,it follows that $|D| < |S \cup \{w\}| \cup |K|$ and hence $\gamma_{cbs}(G) < \gamma_t(G) + \beta_O(G)$.

Equality holds if $G = P_4$,$\gamma_{cbs}(G) = 4, \gamma_t(G) = 2, \beta_0(G) = 2$ gives $\gamma_{cbs}(G) = \gamma_t(G) + \beta_O(G)$.

The following result is a relationship between$\gamma_{cbs}(G)$, domination and connected domination of G.

Theorem 2.8 : Let G be a connected (p, q) graph, then$\gamma_{cbs}(G) + \gamma(G) \geq \gamma_c(G)$.

Proof :Let $C = \{v_1, v_2, v_3, \dots \dots \dots . v_n\} \subseteq V(G)$ be the set of all non end vertices of G.Then $C^1 \subseteq C$ forms a dominating set of G.Let $D \subseteq V(G)$ be a minimal dominating set in G.If $\langle D \rangle$ is connected, then $|D| = \gamma_c(G)$.Otherwise atleast one vertex $\{v_i\} \in V(G)$ with $\deg(v_i) \geq 2$ and $\{D\} \cup \{v_i\} = D_1$ such that $\forall v_m \in V(G) - D_1$ is adjacent to atleast one vertex of $\{D\} \cup \{v_i\}$ and $\langle D_1 \rangle$ is connected. Hence $|D_1| = \gamma_c(G)$.

Let $F = \{v_1, v_2, v_3, \dots \dots \dots . v_n\} \subseteq V[BS(G)]$ which are end vertices which corresponds to the end blocks of G.Let $F_1 = \{v_1, v_2, v_3, \dots \dots \dots . v_m\} \subseteq V[BS(G)]$ which are non end vertices having degree ≥ 2. Suppose $F_1^1 \subseteq F_1$ and $\{F\} \cup \{F_1^1\} = M$ such that $V[BS(G)] - M$ has no isolates. Then $|M| = \gamma_{cbs}(G)$.

Next the following lower bound for cototal domination in $[BS(G)]$ in terms of vertices of G.

Theorem 2.9: For any (p, q) tree T, then $\gamma_{cbs}(T) \geq \left\lceil \frac{P}{2} \right\rceil$ .Equality holds for a pathP_2.

Proof: Let $D = \{u_1, u_2, u_3, \dots \dots \dots . u_n\} \subseteq V[BS(G)]$ be the minimal cototal dominating set of $[BS(G)]$. Suppose $|V[BS(G)] - D| = 0$,then the result follows immediately. Further, if $|V[BS(G)] - D| \geq 2$, then $V[BS(G)] - D$ contains atleast two vertices such that $2n \geq P$. Clearly, it follows that $\gamma_{cbs}(T) = n > \left\lceil \frac{P}{2} \right\rceil$.

Equality holds for a path P_2 , $\gamma_{cbs}(T) = 2, P = 3\ vertices, \left\lceil \frac{P}{2} \right\rceil = 2$.

The following upper bound for cototal domination in $[BS(G)]$ interms of blocks, vertices, domination and cototal domination in cubic graph G.

Theorem 2.10: For any (p, q)separable cubic graph G with $n - blocks$ with $n \geq 2$, then $\gamma_{cbs}(G) \leq n - 1$.

Proof: Suppose G is non separable cubic graph .Then $\gamma_{cbs}(G)$ does not exists. Hence G is a separable graph with $n \geq 2$ blocks.

Let $M = \{B_1, B_2, B_3, \dots \dots \dots B_m\}$ be the set of blocks in G.Suppose $M_1 = \{B_1, B_2, B_3, \dots \dots \dots B_i\}$ be the blocks which are edges in G and $M_2 = \{B_1, B_2, B_3, \dots \dots \dots B_j\}$ be the blocks which not edges,such that $\{M_1 \cup M_2\} = M$. In $S(G)$ every $B_j \in M_1$, are subdivided increases the number of blocks in $S(G)$.

Let $H = \{v_1, v_2, v_3, \dots \dots \dots . v_n\} \subseteq V[BS(G)]$ which are end vertices in $[BS(G)]$, which corresponds to the end blocks of G. Again $H_1 = \{v_1, v_2, v_3, \dots \dots \dots . v_s\} \subseteq V[BS(G)]$ which are non end vertices in $[BS(G)]$, which corresponds to non end blocks in$S(G)$.The distance between (a, b) where $a \in H$ and $b \in H_1$ are at a distance three. Clearly $\{H \cup H_1\}$ is a $\gamma_{cbs}(G)$ −set.Therefore $|H \cup H_1| = \gamma_{cbs}(G)$

Hence $|H \cup H_1| \leq n - 1$ gives $\gamma_{cbs}(G) \leq n - 1$.

Theorem 2.11: For any (p,q) separable cubic graph G, then $\gamma_{cbs}(G) \leq \frac{P}{2} - \delta(G)$.

Proof: Let $|P|$ be the number of vertices in G,Since G is cubic graph, then $\delta(G) = 3$. Suppose $[BS(G)] = M$ be a block subdivision graph of a graph G.Let $H = \{B_1, B_2, B_3, \dots \dots \dots B_n\}$ be the blocks of G and $H_1 = \{B_1^1, B_2^1, B_3^1, \dots\dots B_n^1\}$ be the blocks of $S(G)$.Let $B = \{v_1, v_2, v_3, \dots \dots \dots . v_n\}$ be the set of vertices of M which corresponds to the blocks of H_1. Suppose $B_1 = \{v_1, v_2, v_3, \dots \dots \dots . v_s\}$, $1 \leq s \leq n$,$B_1 \subseteq B$ which are end vertices corresponds to the end blocks of G.Let $B_2 \subseteq V[BS(G)]$ which are non end vertices of B which corespondes to non end blocks of subdivision of G,then $V[BS(G)] - B_2 = J$.Thus $\langle J \cup B_1 \rangle$ has no isolated vertices, which gives the minimum cardinality of a cototal dominating set given by $|J \cup B_1|$.Clearly $|J \cup B_1| \leq \left|\frac{P}{2}\right| - |\delta(G)|$ gives $\gamma_{cbs}(G) \leq \frac{P}{2} - \delta(G)$.

Theorem 2.12: For any (p,q)separable cubic graph, then $\gamma_{cbs}(G) - 1 \leq \gamma(G)$.

Proof: Let $V(G) = \{v_1, v_2, v_3, \dots \dots \dots . v_n\}$ be the set of vertices in G and $V_1(G) = \{v_1, v_2, v_3, \dots \dots \dots . v_s\} \subseteq V(G)$, $1 \leq v_n \leq n$.Then $\langle V(G) - V_1(G) \rangle$=H. Every vertex in $V_1(G)$ is adjacent to atleast one vertex in H.Then H is a $\gamma - set$. Clearly $|H| = \gamma(G)$.

Suppose $M \subseteq V[BS(G)]$ which are end vertices in $B[S(G)]$ which correspondes to the end blocks of G.Let $D \subseteq V[BS(G)]$ be a set , all vertices in $B[S(G)]$ is adjacent to atleast one vertex in D,then $D$is a dominating set .Suppose there exists atleast one vertex $\{v_i\} \in V[BS(G)]$ such that $\langle D \cup \{v_i\} \rangle$ has no isolates. Then $D \cup \{v_i\} = \gamma_{cbs}(G)$. Hence $|D \cup \{v_i\}| - 1 \leq |H|$ gives $\gamma_{cbs}(G) - 1 \leq \gamma(G)$.

Theorem 2.13: For any (p,q) separable cubic graph G, then $\gamma_{cbs}(G) - 2 \leq \gamma_{cot}(G)$.

Proof: Suppose $S = \{v_1, v_2, v_3, \dots \dots \dots . v_n\} \subseteq$ V(G) be the minimal set of vertices which covers all the vertices in G such that $\langle V(G) - S \rangle$ has no isolates. Then $|S|$=$\gamma_{cot}(G)$.

Now without loss of generality in $B[S(G)]$. Let $D = \{u_1, u_2, u_3, \dots \dots \dots . u_k\} \subseteq$ V[BS(G)] be the minimum set of vertices such that for every $x \in N(u)$, where $x \in V[BS(G)] - D$, $u \in D$ and $N(x) \neq \emptyset$ in V[BS(G)] − D. Clearly, D forms a minimal cototal dominating set of $B[S(G)]$. Hence it follows that $|D| - 2 \leq |S|$. Thus $\gamma_{cbs}(G) - 2 \leq \gamma_{cot}(G)$.

Finally, the following result gives a lower bound on $\gamma_{cbs}(G)$ in terms of end blocks in cubic graph G.

Theorem2.14: For any (p,q) separable cubic graph G with N −end blocks , then $\gamma_{cbs}(G) \geq N$.

Proof : Suppose $B[S(G)] = M$ be a block subdivision graph of a graph G. Let $H = \{B_1, B_2, B_3, \dots \dots \dots B_n\}$ be the blocks of G and $H_1 = \{B_1^1, B_2^1, B_3^1, \dots \dots \dots . . B_n^1\}$ be the blocks of $S(G)$.Let $B = \{v_1, v_2, v_3, \dots \dots \dots . v_n\}$ be the set of vertices of M which correspondes to the blocks of H_1 . Suppose $B_1 = \{v_1, v_2, v_3, \dots \dots \dots . v_s\} \subseteq$ V[BS(G)] $\forall\ 1 \leq v_s \leq n$ which are end vertices corresponds to end block of G. Also $B_1^1 \subseteq V[BS(G)]$,$\forall \{v_i\} \in B_1^1$ have degree ≥ 2,then the set$\{V[BS(G)] - B_1 - B_1^1\} = J$, where $\forall \{v_s\} \in J$. Clearly $\langle M \cup J \rangle$has no isolated vertices. Thus $|M \cup J|$ has a minimum cototal domination number. Hence $|M \cup J| \geq N$gives $\gamma_{cbs}(G) \geq N$.

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