3-PARA CONTACT CR– SUBMANIFOLD OF A PARA-SASAKIAN 3-STRUCTURE MANIFOLD

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ABSTRACT

In this paper, we have defined 3-para contact CR-submanifold of a Para-Sasakian 3-Structure manifold. The properties of a Para-Sasakian 3-strcture manifold are obtained and the necessary conditions of a submanifold of it, to be a 3-para-contact CR-submanifold are discussed.

Key words:Differentiable manifold, vector field, 1-form, Riemannian metric, submanifold, distribution.

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INTRODUCTION

Definition (1.1): Let $\tilde{M}$ be a differentiable manifold of dimension *n* (odd). Let there exist a tensor F of type (1, 1), a vector field U a 1-form u and a Riemannian metric g satisfying for arbitrary vector fields $X, Y, Z \ldots$ tangent to $\tilde{M}$,

(1.1) (a) $F^2X = X - u(X)U,$

(b) $u(U) = 1,$

(c) $g(FX, FY) = g(X, Y) - u(X)u(Y),$

(d) $u(X) = g(X, U),$

(e) $(\tilde{\nabla}_X F)(Y) = u(Y)X + g(X, Y)U - 2u(X)u(Y)U,$

then (M, F, U, u, g) is called a Para-Sasakian manifold and U is called a Para-Sasakian structure. For a Para-Sasakian manifold, we easily get

(f) $FU = 0, F^3X = FX, u(FX) = 0, \tilde{\nabla}_X U = -FX,$ $g(FX, Y) = g(X, FY).$

Let us put

(g) $`F(X, Y) = g(FX, Y)$, then we have

(h) $`F(X, Y) = g(FX, Y) = g(X, FY) = g(FY, X) = `F(Y, X),$

Definition (1.2): Let $\underset{x}{U}$ be the Para-Sasakian structure, $\underset{x}{F}$ be the corresponding tensor of type (1,1) $\overset{x}{u}$ be the corresponding 1-form of a Para-Sasakian manifold, where *x*=1,2,3. Then $(\underset{x}{F}, \underset{x}{U}, \overset{x}{u})$, *x*=1,2,3, is called Para-Sasakian 3-structure, if they satisfy for arbitrary X,Y,Z... $\in \tilde{M}$.

(1.2) (a) $\underset{x}{F}\,\underset{y}{F} = \varepsilon_{xyz}\underset{z}{F} + \delta_{xy}I - \overset{y}{u} \otimes \underset{x}{U},$

(b) $\underset{x}{F}\,\underset{y}{U} = \varepsilon_{xyz}\underset{z}{U},$

(c) $\overset{x}{u} \, o \, \underset{y}{F} = \varepsilon_{xyz} \overset{z}{u}$

(d) $\overset{x}{u} \; \underset{y}{U} = \delta_{xy}$

(e) $g \; \underset{x}{F} X, \underset{y}{F} Y = \varepsilon_{xyz} \, g \; \underset{z}{F} X, Y - \overset{x}{u} \; X \; \overset{y}{u} \; Y + \delta_{xy} \, g \; X, Y$

(f) $g \; \underset{x}{U}, \underset{y}{U} = \delta_{xy}$

(g) $\tilde{\nabla}_X \underset{x}{F} \; Y = \overset{x}{u} \; Y \; X + g \; X, Y \; \underset{x}{U} - 2 \overset{x}{u} \; X \; \overset{x}{u} \; Y \; \underset{x}{U}$

where δ_{xy} is 1 if x=y and 0 otherwise, $\varepsilon_{xyz} =$ 1 or -1 according as xyz is an even or odd permutation of 123 and 0 otherwise.

2 SOME SIMPLE PROPERTIES OF PARA-SASAKIAN 3-STRUCTURE:

Theorem (2.1): Let $\underset{x}{F}, \underset{x}{U}, \overset{x}{u}$, x=1,2,3 be a Para-Sasakian 3-structure, then

(2.1) (a) $\underset{r}{F} \underset{r}{U} = 0, \qquad r = 1, 2, 3.$

(b) $\overset{r}{u} \circ \underset{r}{F} = 0, \qquad r = 1, 2, 3.$

(c) (i) $\underset{3}{F} \underset{2}{U} = - \underset{2}{F} \underset{3}{U} = - \underset{1}{U},$

(ii) $\underset{1}{F} \underset{3}{U} = - \underset{3}{F} \underset{1}{U} = - \underset{2}{U},$

(iii) $\underset{2}{F} \underset{1}{U} = - \underset{1}{F} \underset{2}{U} = - \underset{3}{U},$

(d) $F_r^2 = I - \overset{r}{u} \otimes \underset{r}{U}, \qquad r = 1, 2, 3.$

(e) (i) $\underset{2}{F} \underset{3}{F} = \underset{1}{F} - \overset{3}{u} \otimes \underset{2}{U},$

(ii) $\underset{3}{F} \underset{1}{F} = \underset{2}{F} - \overset{1}{u} \otimes \underset{3}{U},$

(iii) $\underset{1}{F} \underset{2}{F} = \underset{3}{F} - \overset{2}{u} \otimes \underset{1}{U},$

(iv) $\underset{3}{F}\,\underset{2}{F} = -\underset{1}{F} - \overset{2}{u} \otimes \underset{3}{U},$

(v) $\underset{1}{F}\,\underset{3}{F} = -\underset{2}{F} - \overset{3}{u} \otimes \underset{1}{U},$

(vi) $\underset{2}{F}\,\underset{1}{F} = -\underset{3}{F} - \overset{1}{u} \otimes \underset{2}{U},$

(f) (i) $\overset{2}{u}\, o\, \underset{1}{F} = -\overset{1}{u}\, o\, \underset{2}{F} = -\overset{3}{u}\,,$

(ii) $\overset{3}{u}\, o\, \underset{2}{F} = -\overset{2}{u}\, o\, \underset{3}{F} = -\overset{1}{u}\,,$

(iii) $\overset{1}{u}\, o\, \underset{3}{F} = -\overset{3}{u}\, o\, \underset{1}{F} = -\overset{2}{u}\,,$

(g) (i) $\tilde{\nabla}_X F_1 \;\; Y = \overset{1}{u} \;\; Y \;\; X + g \;\; X, Y \;\; \underset{1}{U} - 2\overset{1}{u} \;\; X \;\; \overset{1}{u} \;\; Y \;\; \underset{1}{U},$

(ii) $\tilde{\nabla}_X F_2 \;\; Y = \overset{2}{u} \;\; Y \;\; X + g \;\; X, Y \;\; \underset{2}{U} - 2\overset{2}{u} \;\; X \;\; \overset{2}{u} \;\; Y \;\; \underset{2}{U},$

(iii) $\tilde{\nabla}_X F_3 \;\; Y = \overset{3}{u} \;\; Y \;\; X + g \;\; X, Y \;\; \underset{3}{U} - 2\overset{3}{u} \;\; X \;\; \overset{3}{u} \;\; Y \;\; \underset{3}{U},$

Proof (2.1)(a): From (1.2)(b), taking x=1, y=1 $\underset{1}{F}\,\underset{1}{U} = \varepsilon_{11z}\,\underset{z}{U} = 0,$ since 11z is neither even nor an odd permutation of 123 for each value of z=1,2,3 etc.

Proof (2.1)(d): From (1.2(a), taking x=y-1,

$$\underset{1}{F}\,\underset{1}{F} = \varepsilon_{11z}\,\underset{z}{F} + \delta_{11} I - \overset{1}{u} \otimes \underset{1}{U} \qquad \Rightarrow F_1^2 = I - u^1 \otimes \underset{1}{U}, etc.$$

Proof (2.1)(e): From (1.2) (a), taking x=2, y=3,

$$\underset{2}{F}\,\underset{3}{F} = \varepsilon_{23z}\,\underset{z}{F} + \delta_{23} I - \overset{3}{u} \otimes \underset{2}{U} = \underset{1}{F} - \overset{3}{u} \otimes \underset{2}{U},$$

Since 23z is an even permutation of 123 for z=1, etc.

Proof to (2.1) f: From (1.2) (c) taking

$$\overset{2}{u} \circ F_1 = \varepsilon_{21z}\,\overset{z}{u} = -\overset{3}{u},$$

Since 21z is an odd permutation of 123 for z=3 etc.

Proof to (2.1)(g): From (1.2)(g), taking x=1,

$$\nabla_X F_1 \; Y = \overset{1}{u} \; Y \quad X + g \quad X, Y \; \underset{1}{U} - 2\overset{1}{u} \; X \; \overset{1}{u} \; Y \; \underset{1}{U} \; etc$$

3. THE SUBMANIFOLD OF A PARA-SASAKIAN 3-STRUCTURE MANIFOLD

Let *M* be a submanifold of a Para-Sasakian 3-structure manifold $\tilde{M}$. Let *M* be tangent to the structure $\underset{1}{U}, \underset{2}{U}, \underset{3}{U}$. Let *X* be a vector field tangent to M.

We put

(3.1) $\underset{r}{F} X = \underset{r}{P} X + \underset{r}{Q} X, \qquad r = 1,2,3.$

for a vector *N*, normal to *M*

(3.2) $\underset{r}{F} N = t_r N + \underset{r}{f} N, \qquad r = 1,2,3.$

where $t_r N$ and $f_r N$ are the tangential and normal parts respectively.

Theorem (3.1): Let *M* be a submanifold of a Para-Sasakian 3-structure manifold $\tilde{M}$, then

(3.3) (a) $P_r^2 = I - \overset{r}{u} \otimes \underset{r}{U} - t_r Q_r, \qquad r = 1,2,3$

(b) $Q_r \; P_r + f_r Q_r = 0, \qquad r = 1,2,3,$

(c) (i) $P_2 P_1 = - P_3 - \overset{1}{u} \otimes \underset{2}{U} - t_2 Q_1,$

(ii) $P_3 P_2 = - P_1 - \overset{2}{u} \otimes \underset{3}{U} - t_3 Q_2,$

(iii) $P_1 P_3 = - P_2 - \overset{3}{u} \otimes \underset{1}{U} - t_1 Q_3,$

(iv) $P_1 P_2 = P_3 - \overset{2}{u} \otimes \underset{1}{U} - t_1 Q_2,$

(v) $P_2 P_3 = P_1 - \overset{3}{u} \otimes \underset{2}{U} - t_2 Q_3,$

(vi) $P_3 P_1 = P_2 - \overset{1}{u} \otimes \underset{3}{U} - t_3 Q_1.$

(d) (i) $Q_2 P_1 = - Q_3 - f_2 \; Q_1$

(ii) $Q_3 P_2 = - Q_1 - f_3 \; Q_2$

(iii) $Q_1 P_3 = -Q_2 - f_1 Q_3$

(iv) $Q_1 P_2 = Q_3 + f_1 Q_2$

(v) $Q_2 P_3 = Q_1 + f_2 Q_3$

(vi) $Q_3 P_1 = Q_2 + f_3 Q_1$

(e) $f_r^2 = I - Q_r t_r,\ r = 1, 2, 3,$

(f) $P_r t_r + t_r f_r = 0, \quad r = 1, 2, 3,$

(g)(i) $f_2 f_1 = -f_3 - Q_2 t_1$,

(ii) $f_3 f_2 = -f_1 - Q_3 t_2$,

(iii) $f_1 f_2 = -f_2 - Q_1 t_3$,

(iv) $f_1 f_2 = f_3 - Q_1 t_2$,

(v) $f_2 f_3 = f_1 - Q_2 t_3$,

(vi) $f_3 f_1 = f_2 - Q_3 t_1$,

(h) (i) $P_2 t_1 + t_2 f_1 = -t_3$,

(ii) $P_3 t_2 + t_3 f_2 = -t_1$,

(iii) $P_1 t_3 + t_1 f_3 = -t_2$,

(iv) $P_1 t_2 + t_1 f_2 = t_3$,

(v) $P_2 t_3 + t_2 f_3 = t_1$,

(vi) $P_3 t_1 + t_3 f_1 = t_2$,

(i) $P_r \underset{r}{U} = 0 = Q_r \underset{r}{U}, \qquad r = 1,2,3.$

(j)(i) $P_3 U_2 + Q_3 U_2 = -P_2 U_3 - Q_2 U_3 = -U_1$

(ii) $P_1 U_3 + Q_1 U_3 = -P_3 U_1 - Q_3 U_1 = -U_2$

(iii) $P_2 U_1 + Q_2 U_1 = -P_1 U_2 - Q_1 U_2 = -U_3$

(k) (i) $\overset{2}{u} P_1 = -\overset{1}{u} P_2 = -\overset{3}{u}$,

(ii) $\overset{3}{u} P_2 = -\overset{2}{u} P_3 = -\overset{1}{u}$,

(iii) $\overset{1}{u} P_3 = -\overset{3}{u} P_1 = -\overset{2}{u}$.

Proof (3.3)(a),(b): Applying F_r on the left of (3.1)

$$F_r^2 X = F_r \left(P_r X + Q_r X \right) \Rightarrow F_r^2 X = F_r P_r X + F_r Q_r X$$

(3.4) $F_r^2 X = P_r P_r X + Q_r P_r X + t_r Q_r X + F_r Q_r X$

Using (2.1) (d) in (3.4),

(3.5) $X - \overset{r}{u}(X) \underset{r}{U} = P_r^2 X + Q_r P_r X + t_r Q_t X + f_r Q_r X$.

Equating tangential parts on both the sides

$$X - \overset{r}{u}(X) \underset{r}{U} = P_r^2 X + t_r Q_r X \Rightarrow I - \overset{r}{u} \otimes U_r = P_r^2 + t_r Q_r$$

$$\Rightarrow P_r^2 = I - \overset{r}{u} \otimes \underset{r}{U} - t_r Q_r, \qquad r = 1, 2, 3.$$

which is (3.3) (a)

Equating normal parts on both the sides of (3.5),

$$0 = Q_r P_r + f_r Q_r, \qquad r = 1, 2, 3$$

Which is (3.3) (b) Similarly, other parts follow.

4. 3-PARA CONTACT CR-SUBMANIFOLD OF A PARA-SASAKIAN 3-STRUCTURE MANIFOLD:

Definition (4.1): Let M be a submanifold of a Para-Sasakian 3-structure manifold $\tilde{M}$. Let $\underset{1}{U}, \underset{2}{U}, \underset{3}{U}$ be the 3-structures on $\tilde{M}$. M is called a 3-para contact CR-submanifold of $\tilde{M}$ if

(4.1)(a) M is tangent to $\underset{1}{U}, \underset{2}{U}, \underset{3}{U}$

(b) (i) There exists a differentiable distribution

$D : x \longrightarrow D_x \subseteq T_x(M)$, such that

$F_1 D_x \subseteq D_x, F_2 D_x \subseteq D_x, F_3 D_x \subseteq D_x, \forall x \in M$.

(ii) The complementary orthogonal distribution

$D^{\perp} : x \longrightarrow D_x^{\perp} \subseteq T_x(M)$, such that

$F_1 D_x^{\perp} \subseteq T_x^{\perp}\ M\ , F_2 D_x^{\perp} \subseteq T_x^{\perp}\ M\ , F_3 D_x^{\perp} \subseteq T_x^{\perp}\ M\ , \quad \forall\ x \in M$, [ ]

Remark: If M is a 3- para contact CR-submanifold of a Para-Sasakian 3-structure manifold $\tilde{M}$, then $\underset{1}{U}, \underset{2}{U}, \underset{3}{U} \in D.$

Theorem (4.1): Let M be a submanifold of a Para-Sasakian 3-structure manifold $\tilde{M}$. If M is 3-Para contact CR-submanifold of $\tilde{M}$ then

(4.2) $Q_r P_r = 0, \qquad r = 1,2,3$

(4.3) $t_2 Q_1 = 0,\ t_3 Q_2 = 0,\ t_1 Q_3 = 0, t_1 Q_2 = 0,\ t_2 Q_3 = 0, t_3 Q_1 = 0$

Proof: Let us assume that the submanifold M is a 3-Para contact CR-submanifold of Para-Sasakian 3-structrue manifold $\tilde{M}$, . Then as per assumption, (4.1) holds. Therefore there exists a differentiable distribution D and its orthogonal complement $D^{\perp}$.

Let l and $l^{\perp}$ be the projection operators on D and $D^{\perp}$ respectively, then

(4.4) $l + l^{\perp} = I, l^2 = l, l^{\perp 2} = l^{\perp}, ll^{\perp} = l^{\perp} l = 0.$ From (3.1), replacing X by lX,

(4.5) $F_1 lX = P_1 lX + Q_1 lX, F_2 lX = P_2 lX + Q_2 lX, F_3 lX = P_3 lX + Q_3 lX,$

As, $lX \in D$ and D is invariant under $F_1, F_2, F_3;$ gives $F_1 lX,\ F_2 lX,\ F_3 lX \in D$ also $Q_1 l\ X,\ Q_2 l\ X, Q_3 l\ X \in T^{\perp}\ M$ and $P_r lX \in D$ for r= 1, 2, 3. Therefore

(4.6) $l^{\perp} P_r l = 0, Q_r l = 0, r = 1, 2, 3$ From (3.1), replacing X by $l^{\perp} X$,

(4.7) $F_1 l^{\perp} X = P_1 l^{\perp} X + Q_1 l^{\perp} X; F_2 l^{\perp} X = P_2 l^{\perp} X + Q_2 l^{\perp} X; F_3 l^{\perp} X = P_3 l^{\perp} X + Q_3 l^{\perp} X,$

Equating tangential parts on both the sides of (4.7),

(4.8) $P_r l^{\perp} = 0,\ r = 1, 2, 3 \Rightarrow P_r l = P_r, r = 1, 2, 3.$ From (3.3)(b)

(4.9) $Q_r P_r l + f_r Q_r l = 0, r = 1, 2, 3.$ From (4.8) and (4.9)

(4.10) $Q_r P_r + f_r Q_r l = 0, r = 1, 2, 3.$

Using second equation of (4.6) in (4.10), we get

$Q_r P_r = 0,$ which is the required equation (4.2). Operating operating $l^{\perp}$ on the right of each equation of (3.3) (c) in view of (4.8), we get

$t_2Q_1 = 0, t_3Q_2 = 0, t_1Q_3 = 0, t_1Q_2 = 0, t_2Q_3 = 0, t_3Q_1 = 0,$ which is the required equation (4.3)

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