Convolution Theorem for Distributional Fourier-Laplace Transform

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Abstract : *In the tremendous expanding knowledge of science, mathematics plays a vital role. In the words of Philip, Mathematics is a science of quantity and space. Especially in quantum field theory, field of partial differential equations, Harmonic analysis etc. the notion of generalized functions is very essential. The convolution theorem of the transform plays an important role in digital signal processing. The usefulness of convolution theorem can be best explained by its application in filtering.*

This paper is concerned with the generalization of Fourier-Laplace transform in the distributional sense. The main aim of this paper is to prove the properties of convolution and Convolution theorem for Fourier-Laplace transform.

Keywords: Fourier transform, Laplace transform, Fourier- Laplace transform, Integral Transform, Convolution.

1. Introduction

In mathematics and, in particular, functional analysis, convolution is a mathematical operation on two functions f and g, providing a third function that is typically viewed as a modified version of one of the original functions, giving the area overlap between the two functions as a function of the amount that one of the original functions is translated. Convolution is similar to cross-correlation. It has applications that include probability, statistics, computer vision, natural language processing, image and signal processing, electrical engineering and differential equations. Generalizations of convolution have applications in the field of numerical analysis and numerical linear algebra and in the design and implementation of finite impulse response filters in signal processing.

The integral transforms play an important role in signal processing. Fourier analysis is one of the most frequently used tools in signal processing and many other scientific disciplines [1]. The Laplace transform has been widely used in mathematical physics and applied mathematics. It is used to solve the fractional kinetic equations and thermonuclear equations, partial differential equations and integro-differential equations. V.D. Sharma and P.B. Deshmukh [2] had described the convolution theorem for the two-dimensional Fractional Fourier transform in Generalized sense Motivated by the work of V.D. Sharma et.al. Here we discuss the properties of convolution and convolution theorem for Fourier-Laplace transform which is very applicable. So we introduce this Fourier-Laplace transform as follows: $FL\left\{f(t,x)\right\}=F(s,p)=\int_{-\infty}^{\infty}\int_{0}^{\infty}K(t,x)f(t,x)dtdx$,

where, $K(t,x)=e^{-i(st-ipx)}$.

In our previous work we have defined the testing function spaces and Distributional Fourier-Laplace Transform as follows:

1.1. The space $FL_{a,b,\alpha}$

This space is given by

$$FL_{a,b,\alpha}=\left\{\phi:\phi\in E_{+}/\xi_{a,b,k,q,l}\phi(t,x)=\sup_{\substack{0<t<\infty\\0<x<\infty}}\left|t^{k}e^{ax}D_{t}^{l}D_{x}^{q}\phi(t,x)\right|\le C_{lq}A^{k}k^{k\alpha}\right\} \qquad (1.1)$$

Where the constants A and C_{lq} depend on the testing function ϕ.

1.2. The Space $FL_{a,b,\gamma}$

It is given by

$$FL_{a,b,\gamma}=\left\{\phi:\phi\in E_{+}/\gamma_{a,b,k,q,l}\phi(t,x)=\sup_{\substack{0<t<\infty\\0<x<\infty}}\left|t^{k}e^{ax}D_{t}^{l}D_{x}^{q}\phi(t,x)\right|\le C_{lk}A^{q}q^{q\gamma}\right\} \qquad (1.2)$$

Where, $k,l,q=0,1,2,3,\ldots\ldots$ and the constants depend on the testing function ϕ .

1.3. Distributional Generalized Fourier-Laplace Transforms (*FLT*)

For $f(t,x)\in FL_{a,\alpha}^{*\beta}$, where $FL_{a,\alpha}^{*\beta}$ is the dual space of $FL_{a,\alpha}^{\beta}$. It contains all distributions of compact support. The distributional Fourier-Laplace transform is a function of $f(t,x)$ and is defined as

$$FL\left\{f(t,x)\right\}=F(s,p)=\left\langle f(t,x),e^{-i(st-ipx)}\right\rangle , \qquad (1.3.1)$$

where, for each fixed t $(0<t<\infty)$, x $(0<x<\infty)$, $s>0$ and $p>0$, the right hand side of (1.3.1) has a sense as an application of $f(t,x)\in FL_{a,\alpha}^{*\beta}$ to $e^{-i(st-ipx)}\in FL_{a,\alpha}^{\beta}$.

This paper is concerned with the generalization of Fourier-Laplace transform in the distributional sense. In the present paper, Convolution theorem and some properties of convolution is proved

The paper is summarized as follows:

Fourier-Laplace type convolution is given in section 2. Section 3 gives the definition of Fourier-Laplace type convolution. Properties of Fourier-Laplace type convolution are given in section 4. Lastly conclusions are given in section 5.

Notations and terminology as per Zemanian. [3], [4].

2. Fourier-Laplace type convolution

To define Fourier-Laplace type convolution, we require following theorems.

2.1. Theorem

If $f(t,x) \in FL^*_{a,b,\alpha}$ and $\phi(t,x) \in FL_{a,b,\alpha}$ then $\phi \to \psi$ is a continuous linear mapping of $FL_{a,b,\alpha} \to FL_{a,b,\alpha}$, where $\psi(s,p) = \langle f(t,x), \phi(t+s, x+p) \rangle$ (2.1.1)

Proof: By induction method, we can show that

$$D^{m+n}_{s,p}\psi(s,p) = \langle f(t,x), D^{m+n}_{s,p}\phi(t+s,x+p) \rangle$$

For showing $\psi(s,p) \in FL_{a,b,\alpha}$, Consider

$$= \sup_{I_1} \left| s^k e^{ap} D^l_s D^q_p \langle f(t,x), \phi(t+s,x+p) \rangle \right|$$

$$\le C \max_{\substack{0 \le m \le r_1 \\ 0 \le n \le r_2}} \sup_{I_1} \left| s^k e^{ap} D^l_s D^q_p t^k e^{ax} D^m_s D^n_p \phi(t+s,x+p) \right|$$

where r_1 and r_2 are non-negative integers depending on f

$$\le C \max_{\substack{0 \le l+m \le r_1 \\ 0 \le q+n \le r_2}} \sup_{I_1} \left| s^k e^{ap} t^k e^{ax} D^{l+q}_{s,p} D^{m+n}_{s,p} \phi(t+s,x+p) \right|$$

$$\le C \max_{\substack{0 \le l+m \le r_1 \\ 0 \le q+n \le r_2}} \gamma_{a,b,k,l+m,q+n}(\phi) \qquad (2.1.2)$$

Thus, $\psi(s,p) \in FL_{a,b,\alpha}$, continuity follows from (2.1.2) and hence the theorem.

Now we are in a position to define Fourier-Laplace type convolution.

3. Definition

Let a and b be two real numbers with $a \le b$, Fourier-Laplace type convolution is an operation that assigns to each arbitrary choice of the pair $f \in FL^{*}_{a,b,\alpha}$ and $g \in FL^{*}_{a,b,\alpha}$, the Fourier-Laplace type convolution $f * g \in FL^{*}_{a,b,\alpha}$ defined by

$$\left\langle f * g, \phi \right\rangle = \left\langle f(t,x), \left\langle g(s,p), \phi(t+s, x+p) \right\rangle \right\rangle \text{, where } \phi \in FL_{a,b,\alpha} \qquad (3.1)$$

4. Properties of Fourier-Laplace type convolution

4.1. Theorem:

If $f(t,x) \in FL^{*}_{a,b,\alpha}$ and $g \in D_{+}(I)$ then $g \to f * g$ is a continuous, linear mapping from $D_{+}(I)$ into E_{+}, where

$$(f * g)(s,p) = \left\langle f(t,x), g(s-t, p-x) \right\rangle.$$

Proof: It is easy to prove that $(f * g)$ is smooth and the mapping is linear. For its continuity,

$$\left| D^{k_1+k_2}_{s,p} (f * g)(s,p) \right| = \left| \left\langle f(t,x), D^{k_1+k_2}_{s,p} \left\{ g(s-t, p-x) \right\} \right\rangle \right|$$

$$\le C \max_{\substack{0 \le l + k_1 \le r_1 \\ 0 \le q + k_2 \le r_2}} \sup_{I_1} \left| D^{l+q}_{t,x}, D^{k_1+k_2}_{s,p} \left\{ g(s-t, p-x) \right\} \right|$$

Since $g \in D_{+}(I)$, continuity follows from the above inequality. We call

$(f * g)(s,p) = \left\langle f(t,x), g(s-t, p-x) \right\rangle$ as Fourier-Laplace type regularization of g by f.

4.2. Theorem:

Convolution operation in (4.1) commutes with shifting scaling operator S i.e. $S(f * g) = f * (S(g))$.

Proof: Consider

$$\left\langle S(f * g), \phi(t,x) \right\rangle = \left\langle f * g, \phi(t+s, x+p) \right\rangle$$

$$= \left\langle f, \left\langle g, \phi(s-t, p-x) \right\rangle \right\rangle \qquad (4.2.1)$$

Now, $\left\langle f*S(g),\phi(t,x)\right\rangle=\left\langle f,\left\langle S(g),\phi(t+s,x+p)\right\rangle\right\rangle$

$$=\left\langle f,\left\langle g,S\left(\phi(t+s,x+p)\right)\right\rangle\right\rangle$$

$$=\left\langle f,\left\langle g,\phi(s-t,p-x)\right\rangle\right\rangle \qquad (4.2.2)$$

Theorem from (4.2.1) and (4.2.2), we write

$$S(f*g)=f*\left(S(g)\right)$$

4.3. Theorem:

$f\in D_+^*$ and $FL\left\{f(u,v)\right\}=F(s,p)$, s and $p\in\Omega_f$ and if $g\in D_+^*$, $FL\left\{g(t,x)\right\}=G(s,p)$, s and $p\in\Omega_g$ and $\Omega_f\cap\Omega_g$ is not empty, then $f*g$ exist in the sense of FL-type convolution in $FL_{a,b,\alpha}^*$, where the strip of definition is the intersection of $\Omega_f\cap\Omega_g$ with real axis. Moreover, $FL(f*g)=FL(f)\cdot FL(g)$

Proof: Using theorem 4.2 it can be easily shown that $f*g\in FL_{a,b,\alpha}^*$. Further as $K(t,x,s,p)=e^{-i(st-ipx)}\in FL_{a,b,\alpha}$, for each fixed s and p with $a<\operatorname{Re}s<\infty$, $c<\operatorname{Re}p<d$.

$$FL(f*g)=\left\langle f*g,e^{-i(st-ipx)}\right\rangle$$

$$=\left\langle f(u,v),\left\langle g(t,x),e^{-is(t+u)}e^{-p(x+v)}\right\rangle\right\rangle$$

$$=\left\langle f(u,v),e^{-isu}e^{-pv}\right\rangle\left\langle g(t,x),e^{-ist}e^{-px}\right\rangle$$

$$=\left\langle f(u,v),e^{-i(su-ipv)}\right\rangle\left\langle g(t,x),e^{-i(st-ipx)}\right\rangle$$

$$=FL\left\{f(u,v)\right\}\cdot FL\left\{g(t,x)\right\}$$

$$=F(s,p)\cdot G(s,p)$$

Hence the theorem.

5. Conclusion

This paper is concerned with the generalization of Fourier-Laplace transform in the distributional sense. In this paper the Fourier-Laplace type convolution and its properties are proved which will be useful in signal processing, filtering and also solving Integro-differential equations.

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