Higher Dimensional Kaluza-Klein Cosmological Model with Time Dependent G and Λ term in General Relativity

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ABSTRACT

In this paper we have obtained the exact solution of Einstein's field equations with time dependent gravitational and cosmological constant term in the presence of perfect fluid for the five dimensional Kaluza-Klein space-time. The nature of G(t) and Λ (t)have been examined for two different cases, it is observed that due to the combined effect of time dependent Λ and G the universe evolved with decelaration as well as accelaration. We also obtain the value of scalar expansion θ , shear tensor σ_{ij} and the Hubbles parameter H. Moreover it is observed that for this model G varies with time as suggested by large number hypothesis proposed by Dirac. Some physical and geometrical properties of the model are discussed which are in good agreement with the observational results.

Keywords: Cosmology, exact solution, five dimension, perfect fluid and G(t), Λ (t).

1. Introduction

The Einstein field equation has two parameters, the gravitational constant G and the cosmological constant Λ. The Newtonian constant of gravitation G plays the role of a coupling constant between geometry and matter in the Einstein field equation. In an evolving universe, it appears to look at this constant as a function of time. There are significant observational evidence that the expansion of the Universe is undergoing a late time acceleration. This, in other words, amounts to saying that in the context of Einstein's general theory of relativity some sort of dark energy, constant or that varies only slowly with time and space dominates the current composition of cosmos. The origin and nature of such an accelerating field poses a completely open question. The exact physical situation at very early stages of the formation of our universe provoked great interest among researchers. Higher dimensional space-time has taken considerable research interest in an attempt to unify gravity with other forces in nature. This idea is particularly important in the field of cosmology since one knows that our universe was much smaller in its early stage than it is today. In this connection a number of attempts have been made to study the role of gravity with other fundamental forces in nature.

In modern cosmological theories, the cosmological constant plays an important role. A wide range of observations now compellingly suggest that the universe possesses a non-zero cosmological constant. In the context of quantum field theory, a cosmological term corresponds to the energy density of vacuum. The birth of the universe has been attributed to an excited vacuum fluctuation triggering off an inflationary expansion followed by the supercooling. The release of locked up vacuum energy results in subsequent reheating. The cosmological term, which is measure of the energy of empty space, provides a repulsive force opposing the gravitational pull between the galaxies. If the cosmological term exits, the energy it represents counts as mass because mass and energy are equivalent. If the cosmological term is large enough, its energy plus the matter in the universe could lead to inflation. Unlike standard inflation, a universe with a cosmological term would expand faster with time because of the push from the cosmological term.

In recent past, number of researchers have considered cosmological models with time-dependent cosmological constant. Also, numerous modifications of general relativity allowed for a variable G by different arguments have been proposed. It was for the first time that Dicke and Dirac suggested a possible time varying gravitational constant. The Large Number Hypothesis (LNH) proposed by Dirac leads to a cosmology when G varies with time. Variation of G has many interesting consequences in astrophysics. Canuto and Narlikar have shown that G-varying cosmology is consistent with whatsoever cosmological observations available at present. A modification linking the variation of G with that of variable Λ-term has been considered within the framework of general relativity by the number of researchers.

Recently, Ray et al. (2007) obtained dark energy models with time-dependent G. Mukhopadhyay et al. (2007, 2010) studied higher dimensional dark energy with time variable Λ and G. Arbab (2008) investigated bulk viscous dark energy models with variable G and Λ. Pradhan et al. (2011) also obtained FRW universe with variable G and Λ-term. Khadekar and Kamdi (2006, 2007) have obtained exact solution of the Einstein's field equations in higher dimension with variable G and Λ. Singh et al. (2003, 2004, 2007) and Bali et al. (2010) have investigated Bianchi type-III cosmological models with variable G and Λ in presence of perfect fluid and their both solutions are particular and similar. Hasan A. et al. (2011) have studied Bianchi Type-III cosmological model with variable G and Λ-term in general relativity. The aforesaid survey of literature clearly indicates that there has been interest in studying Bianchi types models with variable G and Λ.

The most famous five dimensional theory proposed by Kaluza and Klein, was the first theory in which gravitation and electromagnetism could be unified in a single geometrical structure. The Kaluza-Klein's idea to consider the coefficient of the fifth co-ordinate as constant, was generalized by Thiry and Jordan. Marciano has suggested that the experimental detections of time variations of the fundamental constants could be strong evidence of extra dimensions. To achieve unification of all interactions including weak and strong forces, many authors have extended the Kaluza-Klein formalism to higher dimensions. The investigations of super-string theory and super gravitational theory have created renewed interest among theoretical physicists to study the physics in higher dimensional space-time. Multidimensional space-time is believed to be relevant in the context of cosmology.

Thorough study of Kaluza-Klein theory has been undertaken by Wesson. A number of authors have studied the physics of the universe in higher dimensional space-time.

The aforesaid survey of literature clearly indicates that there has been an interest in studying cosmological models in higher dimensional space-time with variable gravitational and cosmological constant to obtain the exact solution of Einstein's field equations which is new and different from others. Motivated by above studies we consider the five dimensional Kaluza-Klein space-time to obtain the exact solution of Einstein's field equations with time dependent gravitational constant G and cosmological constant Λ-term, which places an important role during the early stages of the universe.

The outline of this paper is as follows: Section [2] describes the metric and field equations. In Section [3], we find the exact solutions of field equations and their geometric and physical properties. Finally conclusions are summarize in the last section [4].

2. The Metric and Field Equations

Consider the five dimensional Kaluza-Klein metric in the form

$$ds^2 = -dt^2 + R^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right) + A^2(t) d\mu^2 \qquad \dots (2.1)$$

Unlike Wesson (1983), the fifth co-ordinate is taken to be space-like and the metric coefficients are assumed to be functions of time only. Here the spatial curvature has been taken as zero (Gron 1988).

The energy momentum tensor for perfect fluid distribution has the form

$$T_i^j = (p+\rho)u_i u^j + p\delta_i^j \qquad \dots (2.2)$$

where u_i is the unit flow vector and satisfies condition

$$u_i u^i = -1 \qquad \dots (2.3)$$

Here p is isotropic pressure, ρ is the proper energy density, in a co-moving coordinate system, we have

$$u^i = (0,0,0,0,1) \qquad \dots (2.4)$$

The Einstein's field equations with time varying G and Λ read as

$$G_i^j = R_i^j - \frac{1}{2}R\delta_i^j = -8\pi G(t)T_i^j + \Lambda(t)\delta_i^j \qquad \dots (2.5)$$

where R_i^j is the Ricci tensor; $R = g^{ij} R_{ij}$ is the Ricci scalar, G is the gravitational constant and Λ is cosmological constant.

The field equations (2.5) with (2.2) for the metric (2.1) subsequently leads to the following system of equations:

$$2\frac{\ddot{R}}{R} + \frac{\ddot{A}}{A} + \frac{\dot{R}^2}{R^2} + 2\frac{\dot{A}}{A}\frac{\dot{R}}{R} + \frac{K}{R^2} = -8\pi G(t)p + \Lambda(\mathrm{t}) \quad \ldots (2.6)$$

$$3\frac{\ddot{R}}{R} + 3\frac{\dot{R}^2}{R^2} + 3\frac{K}{R^2} = -8\pi G(t)p + \Lambda(\mathrm{t}) \quad \ldots (2.7)$$

$$3\frac{\dot{R}^2}{R^2} + 3\frac{\dot{A}}{A}\frac{\dot{R}}{R} + \frac{3K}{R^2} = 8\pi G(t)\rho + \Lambda(\mathrm{t}) \quad \ldots (2.8)$$

where overhead dot denotes derivatives w.r.t. time t.

Vanishing divergence of Einstein tensor

$$G_{i;j}^{j} = 0 \quad \ldots (2.9)$$

$$\Rightarrow 8\pi\dot{G}(t)\rho + \dot{\Lambda}(\mathrm{t}) + 8\pi G(t)\left[\dot{\rho} + \rho\left(3\frac{\dot{R}}{R} + \frac{\dot{A}}{A}\right) + p\left(\frac{\dot{R}}{R} + \frac{\dot{A}}{A}\right)\right] + \Lambda(\mathrm{t})\frac{2\dot{R}}{R} = 0 \quad \ldots (2.10)$$

According to the law of conservation of energy

$$T_{i;j}^{j} = 0 \quad \ldots (2.11)$$

$$\Rightarrow \dot{\rho} + \rho\left(3\frac{\dot{R}}{R} + \frac{\dot{A}}{A}\right) + p\left(\frac{\dot{R}}{R} + \frac{\dot{A}}{A}\right) = 0 \quad \ldots (2.12)$$

Using (2.12) in (2.10), we get

$$\Rightarrow 8\pi\dot{G}(t)\rho + \dot{\Lambda}(\mathrm{t}) + \Lambda(\mathrm{t})\frac{2\dot{R}}{R} = 0 \quad \ldots (2.13)$$

The spatial volume for the metric (1) is given by

$$V^3 = \frac{r^2 sin\theta}{\sqrt{1-kr^2}} AR^3 \quad \ldots (2.14)$$

$$\Rightarrow V = \left[\frac{r^2 sin\theta}{\sqrt{1-kr^2}}\right]^{1/3} A^{1/3} R \quad \dots (2.15)$$

Therefore the Hubble's parameter is given by

$$H = \frac{\dot{V}}{V} = \frac{1}{3}\left[3\frac{\dot{R}}{R} + \frac{\dot{A}}{A}\right] \quad \dots (2.16)$$

Hence the generalized mean Hubble's parameter is defined by

$$H = \frac{1}{3}(H_0 + H_1 + H_2 + H_3) \quad \dots (2.17)$$

where $H_0 = H_1 = H_2 = \frac{\dot{R}}{R}$ and $H_3 = \frac{\dot{A}}{A}$ ... (2.18)

are the directional Hubble's parameters in the directions of r, θ, ϕ and μ respectively.

The scalar expansion θ is defined by

$$\theta = \frac{1}{3}u^i_{;i} = \frac{1}{3}\left[u^i_{;i} + u^k \Gamma^i_{ki}\right] \quad \dots (2.19)$$

$$\Rightarrow \theta = \frac{1}{3}\left[3\frac{\dot{R}}{R} + \frac{\dot{A}}{A}\right] \quad \dots (2.20)$$

The components of Shear σ_{ij} are defined by

$$\sigma_{ij} = \frac{1}{2}\left[u_{i;j} + u_{j;i}\right] + \frac{1}{2}\left[u_{i;k}u^k + u_i u_{j,k} u^k\right] - \frac{1}{3}\theta\left[g_{ij} - u_i u_j\right]$$

$$\sigma^2 = \frac{4}{9}\theta^2 = \frac{4}{81}\left[3\frac{\dot{R}}{R} + \frac{\dot{A}}{A}\right]^2 \quad \dots (2.21)$$

Deceleration parameter q is defined as

$$q = -\frac{3}{\theta^2}\left[\theta_{;\alpha}u^\alpha + \frac{\theta^2}{3}\right] \quad \dots (2.22)$$

3. Exact Solutions of Field Equations

The Einstein's field equations (2.6) - (2.8) are a system of three equations with six unknowns $R, A, p, \rho, G, \Lambda$.

Therefore, hereweassume the relation $A = R^n$ because of the fact that the field equations are highly non-linear.

Using this relation, in the field equations, we get

$$R = C_1 t + C_2 = T \quad \dots (3.1)$$

where C_1 and C_2 are constants of integration.

Then $A = R^n = T^n$ ... (3.2)

Hence metric (2.1) becomes

$$\mathrm{ds}^2 = -\mathrm{dt}^2 + \mathrm{T}^2\left(\frac{\mathrm{dr}^2}{1-\mathrm{kr}^2} + \mathrm{r}^2\mathrm{d}\theta^2 + \mathrm{r}^2\sin^2\theta\mathrm{d}\phi^2\right) + \mathrm{T}^{2\mathrm{n}}\mathrm{d}\mu^2 \quad \dots (3.3)$$

Spatial volume: $V^3 = \sqrt{-g} = \frac{T^{3+n} r^2 sin\theta}{\sqrt{1-kr^2}}$ ... (3.4)

Scalar expansion: $\theta = \frac{1}{3} u^i_{;i} = -\frac{3+n}{3T}$ ... (3.5)

Deceleration parameter: $q = -\left[\frac{n-6}{n+3}\right] > 0$ for $-3 < n < 6$ ... (3.6)

Shear Scalar: $\sigma^2 = \sigma_{ij}\sigma^{ij} = \frac{4(n+3)^2}{81T^2}$ ... (3.7)

Thus, volume increases with time and since $q > 0$, therefore the model is not inflationary.

But Shear decreases as time increases and since $\lim_{T\to\infty} \frac{\sigma}{\theta} \neq 0$, the model is anisotropic for all values of T.

Now we assume that the fluid obeys the barotropic equation of state

$$p = \gamma\rho \quad \dots (3.8)$$

where γ is a constant with $0 < \gamma < 1$.

Using (3.1), (3.2) and (2.12) in (2.10), we get

$$\rho = \frac{C_3}{T^{n(r+1)+(r+3)}} \quad \dots (3.9)$$

and hence $p = \gamma \frac{C_3}{T^{n(r+1)+(r+3)}}$ ... (3.10)

As equations (2.8) and (2.13) are in terms of G and Λ and their derivatives, it is difficult to get the values of G and Λ independently. Therefore we consider two different cases:

Case I: Several authors are suggested that $\Lambda \propto R^{-2}$, by considering different assumptions. Authors W. Chen and Y. Wu (1990) have suggested that in the relation $\Lambda = \alpha R^{-2}$, the constant α is related to the curvature parameter k and hence we may assume

$$\Lambda = C_4 R^{-2} = \frac{C_4}{T^2} \quad \ldots (3.11)$$

where the constant C_4 is related to the curvature parameter k.

Then from (2.8), we get

$$G = \left\{\frac{3[(n+1)C_1^2+k]-C_4}{8\pi C_3}\right\} T^{(n+1)(\gamma+1)} \quad \ldots (3.12)$$

Case II: In most of the investigations, a power law relation between the scalar factor and scalar fields assumed. Cosmological models with the gravitational and cosmological constants generalized as coupling scalars where $G \sim R^{\alpha}$ discussed by Anirudh Pradhan et.al. (2007).

In this case we shall consider

$$G = C_5 R^{\alpha} = C_5 T^{\alpha} \quad \ldots (3.13)$$

where C_5 is the proportionality constant and $\alpha \geq 0$.

Then from equation (2.8), we obtain

$$\Lambda = \frac{3[(n+1)C_1^2+k]}{T^2} - \frac{8\pi C_3}{T^{\gamma(n+1)+(n+3)-\alpha}} \quad \ldots (3.14)$$

4. Conclusion

We have obtained an exact solution of Einstein's field equation with time varying gravitational constant G and cosmological constant Λ. The behaviour of the universe in this model is determined by the cosmological term Λ, this term has the same effect as a uniform mass density $\rho_{eff} = -\frac{\Lambda}{4\pi G}$, which is constant in space and time. A positive value of Λ corresponds to a negative effect of mass density, i.e. repulsive nature of gravitation. Hence we can predict that in the universe positive value of Λ corresponds to the expansion which will accelerate, whereas negative value of Λ will slow down, stop and reverse the expansion of the universe.

The physical quantities ρ and Λ tends to infinity as $T \to 0$ and tend to zero as $T \to \infty$, whereas the gravitational constant $G \to 0$ as $T \to 0$ and $G \to \infty$ as $T \to \infty$. Thus the model (3.3) starts with a Big-Bang at $T = 0$ and it goes on expanding untill it comes to rest at infinite T, where $T \to 0$ and $T \to \infty$ respectively correspond to the proper time $t \to 0$ and $t \to \infty$.

In our derived model, we have observed that the Λ term decreases as time increases and it approaches to a small and positive value at the present epoch. Thus our models are consistent with the results of recent observations.

Also gravitational constant G(t) increases or decreases as Λ(t) decreases or increases. In our case, G increases as Λ decreases with time. Thus it has been found that all physical and geometrical parameters of the model are in good agreement with the observational results.

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