Operational Calculus on Generalized Two-Dimensional Offset Fractional Fourier Transform

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***Abstract*—Fractional FourierTransform (FRFT) is a linear transform generalized from the conventional Fourier transform. It has much attention in recent years. Many properties of the Fractional Fourier transform are well known in signal community and processing applied into information encryption and image processing. The Offset FRFT is more flexible than the original FRFT.In this paper we present generalization of Two Dimensional Offset Fractional Fourier transform(2D Offset FRFT) in distributional sense. Operational calculus on the generalized Two Dimensional Offset Fractional Fourier Transform is presented**

***Keywords*—Fractional Fourier transform, Two-dimensional Offset Fractional Fourier transform, Two Dimensional Fractional Fourier Transform, Generalized Function.**

I.INTRODUCTION

As a Generalization of Fourier transform, the Fractional Fourier transform (FRFT) contains simultaneity the time-frequency information of the signal and is considered as a new tool for time frequency analysis, especially in the area of image recognition. In the domain of optical, the optical definitions of continuous FRFT, with its optical implementation, fractional convolution and correlation operation have been investigated with possible application proposed [1-3].

FRFT is important tool in sonar signal processing as it takes advantage of the knowledge of transmitted waveform. Another important issue in signal processing is filtering; Filtering in frequency domain is widely employed for its easy implementation and high efficiency in many cases [4, 5]. In the area of facial expression recognition, some researchers have shown its superiority with respect to other feature extraction tools [6]. Two-dimensional FRFT is powerful tool for feature extraction in facial expression recognition. It is known that different order of Two-dimensionalFRFT contain different time frequencyinformation.

Two-dimensional Offset Fractional Fourier transform is the extension of Two-dimensional FRFT. Many applications of FRFT are also potential application of the offset fractional Fourier transform [7].

This paper organized as follows. In section 2 Two Dimensional Offset Fractional Fourier Transform and Testing Function space are defined. In section 3 Generalized 2D Offset FRFT is defined. In section 4 Some properties of the kernel and operation transform formulae on 2D Offset FRFT are proved.

II. DEFINITIONS OF 2D OFFSET FRFT AND TESTING FUNCTION SPACE

A. *Two Dimensional Offset Fractional Fourier Transform*

Two Dimensional Offset Fractional Fourier Transform $\left[F_\alpha^{\tau,\eta,\zeta,\gamma} f(t,x)\right](s,u)$ of function $f\,(t,x)$ through an angle α is defined as

$$\left[F_\alpha^{\tau,\eta,\zeta,\gamma} f(t,x)\right](s,u) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,x) K_\alpha(t,s-\eta,x,u-\gamma).\,dtdx$$

Where,

$$K_\alpha(t,s-\eta,x,u-\gamma) = \sqrt{\frac{1-i\cot\alpha}{2\pi}}\, e^{i(s\tau+u\zeta)}\, e^{\frac{1}{2\sin\alpha}\left[\left(t^2+(s-\eta)^2+x^2+(u-\gamma)^2\right)\cos\alpha-2\left(t(s-\eta)+x(u-\gamma)\right)\right]}$$

B. Testing function space E

An infinitely deferential complex valued smooth function on $\phi(R^n)$ belongs to $E(R^n)$, if for each contact $I \subset S_{a,b}$

Where

$$S_{a,b} = \{t; x : t\, x \in R^n\}\ ; \quad |t| \le a, |x| \le b, a < 0,\ b < 0\}, I \in R^n$$

$$\gamma_{I,l,q}(\phi) = \sup_{t,x,\in I} \left|D_{t,x}^{l,q}\phi(t,x)\right| < \infty \qquad l,q = 0,1,2\ \ldots\ldots\ldots$$

III. Distributional Two-Dimensional Offset Fractional Fourier Transform

The Two-Dimensional Offset Fractional Fourier Transform $\left[F_\alpha^{\tau,\eta,\zeta,\gamma} f(t,x)\right](s,u)$ of generalization function f (t, x) through an angle α is defind as

$$\left[F_\alpha^{\tau,\eta,\zeta,\gamma} f(t,x)\right](s,u) = \langle f(t,x)\, K_\alpha(t, s-\eta, x, u-\gamma)\rangle$$

Where

$$K_\alpha(t, s-\eta, x, u-\gamma) = \sqrt{\frac{1 - i\cot\alpha}{2\pi}}\, e^{i(s\tau+u\zeta)}$$
$$e^{\frac{1}{2\sin\alpha}\left[\left(t^2+(s-\eta)^2+x^2+(u-\gamma)^2\right)\cos\alpha - 2\left(t(s-\eta)+x(u-\gamma)\right)\right]}$$

$$K_\alpha(t, s-\eta, x, u-\gamma) = C_{1\alpha} e^{i(s\tau+u\zeta)} e^{C_{2\alpha}\left[\left(t^2+(s-\eta)^2+x^2+(u-\gamma)^2\right)\cos\alpha - 2\left(t(s-\eta)+x(u-\gamma)\right)\right]}$$

Where $C_{1\alpha} = \sqrt{\frac{1-i\cot\alpha}{2\pi}}$ and $C_{2\alpha} = \frac{1}{2\sin\alpha}$

IV. Properties of Kernel of 2D Offset FRFT:

A. *Result -1*

To Prove

$$K_{(-\alpha)}(t, s-\eta, x, u-\gamma) = K_\alpha^*(t, s-\eta, x, u-\gamma)$$

Where * denotes the conjugation.

Consider

$$K_\alpha(t, s-\eta, x, u-\gamma)$$
$$= \sqrt{\frac{1 - i\cot\alpha}{2\pi}}\, e^{i(s\tau+u\zeta)}$$
$$e^{\frac{i}{2\sin\alpha}\left[\left(t^2+(s-\eta)^2+x^2+(u-\gamma)^2\right)\cos\alpha - 2\left((-t)(s-\eta)+(-x)(u-\gamma)\right)\right]}$$

$$= \sqrt{\frac{1 + i\cot\alpha}{2\pi}}\, e^{i(s\tau+u\zeta)}$$
$$e^{-\frac{i}{2\sin\alpha}\left[\left(t^2+(s-\eta)^2+x^2+(u-\gamma)^2\right)\cos\alpha - 2\left((-t)(s-\eta)+(-x)(u-\gamma)\right)\right]}$$
$$= \sqrt{\frac{1 - (-i)\cot\alpha}{2\pi}}\, e^{i(s\tau+u\zeta)}$$
$$e^{-\frac{i}{2\sin\alpha}\left[\left(t^2+(s-\eta)^2+x^2+(u-\gamma)^2\right)\cos\alpha - 2\left((-t)(s-\eta)+(-x)(u-\gamma)\right)\right]}$$
$$= K_\alpha^*(t, s-\eta, x, u-\gamma)$$

Where * denotes the conjugation.

B. *Result -2*

To Prove

$$K_\alpha(-t, s-\eta, -x, u-\gamma) = \frac{1}{e^2} K_\alpha(t, s-\eta, x, u-\gamma)$$

Consider

$$K_\alpha(-t, s-\eta, -x, u-\gamma)$$
$$= \sqrt{\frac{1 - i\cot\alpha}{2\pi}}\, e^{i(s\tau+u\zeta)}$$
$$e^{\frac{1}{2\sin\alpha}\left[\left(t^2+(s-\eta)^2+x^2+(u-\gamma)^2\right)\cos\alpha - 2\left((-t)(s-\eta)+(-x)(u-\gamma)\right)\right]}$$

$$K_\alpha(-t, s-\eta, -x, u-\gamma)$$
$$= \sqrt{\frac{1 - i\cot\alpha}{2\pi}}\,.\, e^{i(s\tau+u\zeta)} e^{\frac{1}{2\sin\alpha}\left[\left(t^2+(s-\eta)^2+x^2+(u-\gamma)^2\right)\cos\alpha\right]}$$
$$e^{\frac{-i}{2\sin\alpha}\left[2\left((-t)(s-\eta)\right)\right]}\; e^{\frac{-i}{2\sin\alpha}\left[2\left((-x)(u-\gamma)\right)\right]}$$
$$= \frac{1}{e^2} K_\alpha(t, s-\eta, x, u-\gamma)$$

C. *Result -3*

$$K_\alpha(t, 0, x, 0) = e^{\frac{-i}{2\sin\alpha}\left[\left(t^2+(s-\eta)^2+x^2+(u-\gamma)^2\right)\cos\alpha - 2\left((-t)(s-\eta)+(-x)(u-\gamma)\right)\right]}$$
$$K_\alpha(t, s-\eta, x, u-\gamma)$$

Consider

$$K_\alpha(t, s-\eta, x, u-\gamma)$$
$$= \sqrt{\frac{1 - i\cot\alpha}{2\pi}}\, e^{i(s\tau+u\zeta)}$$
$$e^{\frac{i}{2\sin\alpha}\left[\left(t^2+(s-\eta)^2+x^2+(u-\gamma)^2\right)\cos\alpha - 2\left((-t)(s-\eta)+(-x)(u-\gamma)\right)\right]}$$

$$K_\alpha(t, 0, x, 0)$$
$$= \sqrt{\frac{1 - i\cot\alpha}{2\pi}}\, e^{i(s\tau+u\zeta)} e^{\frac{i}{2\sin\alpha}\left[\left(t^2+x^2\right)\cos\alpha\right]}$$

$$K_\alpha(t, 0, x, 0) = \sqrt{\frac{1 - i\cot\alpha}{2\pi}}\, e^{i(s\tau+u\zeta)}$$

$$e^{\frac{1}{2\sin\alpha}\left[\left(t^2+(s-\eta)^2+x^2+(u-\gamma)^2\right)\cos\alpha - 2\left(t(s-\eta)+x(u-\gamma)\right)\right]} e^{\frac{-i}{2\sin\alpha}\left[\left((s-\eta)^2+(u-\gamma)^2\right.\right.}$$
$$= e^{\frac{-i}{2\sin\alpha}\left[\left((s-\eta)^2+(u-\gamma)^2\right)\cos\alpha - 2\left((s-\eta)t+(u-\gamma)x\right)\right]}$$
$$K_\alpha(t, s-\eta, x, u-\gamma)$$

D. Proposition

Generalization of offset 2D Fractional Fourier transform reduces to conventional offset 2D Offset FRFT If $\theta=\frac{\pi}{2}$

2D Offset Fractional Fourier Transform is

$$\left[F_\alpha^{\tau,\eta,\zeta,\gamma} f(t,x)\right](s,u)$$
$$= \langle f(t,x) k_\alpha(t,s-\eta,x,u-\gamma)\rangle \left[F_\alpha^{\tau,\eta,\zeta,\gamma} f(t,x)\right](s,u)$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,x) C_{1\alpha} e^{i(s\tau+u\zeta)} dtdx \; e^{C_{2\alpha}\left[(t^2+(s-\eta)^2+x^2+(u-\gamma)^2)\cos\alpha-2(t(s-\eta)+x(u-\gamma))\right]}$$

$$\left[F_{\frac{\pi}{2}}^{\tau,\eta,\zeta,\gamma} f(t,x)\right](s,u)$$

$$=\sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,x) e^{-i[(t(s-\eta)t+x(u-\gamma))]} dtdx$$
$$= \text{2D Offset FRFT}\{f(t,x)\}$$

V. Operation transforms formulae for 2D offset fractional Fourier transform

A. Linearly property

If $\text{2DOffsetFRFT}\{f(t,x)\}$ and $\text{2D Offset FRFT}\{g(t,x)\}$ is generalized outline Fractional Fourier transform of $f(t,x)$ and $g(t,x)$ then

$$\text{2D OffsetFRFT}\{c_1 f(t,x)+c_2 g(t,x)\}$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[c_1 f(t,x)+c_2 g(t,x)]\, K_\alpha(t,s-\eta,x,u-\gamma).\,dt.\,dx$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c_1 f(t,x) K_\alpha(t,s-\eta,x,u-\gamma).\,dt.\,dx + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c_2 g(t,x) K_\alpha(t,s-\eta,x,u-\gamma).\,dt.\,dx$$
$$= c_1\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,x) K_\alpha(t,s-\eta,x,u-\gamma).\,dt.\,dx + c_2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(t,x) K_\alpha(t,s-\eta,x,u-\gamma).\,dt.\,dx$$
$$= c_1[\text{2D Offset FRFT}(t,x)](s-\eta,u-\gamma) + c_2[\text{2D Offset FRFT}(t,x)](s-\eta,u-\gamma)$$

B. Differential property

Result:

$$\text{2D Offset FRFT}\{f(t,x)\}(s-\eta,u-\gamma) = (-i\cot\alpha)\text{2D Offset FRFT}\{tf(t,x)\}(s-\eta,u-\gamma) + [i(u-\gamma)\operatorname{cosec}\alpha]\text{2D Offset FRFT}\{f(t,x)\}(s-\eta,u-\gamma)$$

Consider

$$\text{2D Offset FRFT}\{f(t,x)\}(s-\eta,u-\gamma)$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c_{1\alpha}\, e^{i(s\tau+u\zeta)}\, e^{\frac{1}{2\sin\alpha}[t^2+(s-\eta)^2+x^2+(u-\gamma)^2]\cos\alpha-2[t(s-\eta)+x(u-\gamma)]} f'(t,x)dt\,dx$$
$$= c_{1\alpha} e^{i(s\tau+u\zeta)} e^{\frac{i}{2}[(s-\eta)^2+(u-\gamma)^2\cot\alpha]}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2}[(t^2+x^2)\cot\alpha-i(t(s-\eta)+x(u-\gamma))]\operatorname{cosec}\alpha} f'(t,x)dx\,dx$$
$$= c_{1\alpha} e^{i(s\tau+u\zeta)} e^{\frac{i}{2}[(s-\eta)^2+(u-\gamma)^2\cot\alpha]}\int_{-\infty}^{\infty} e^{\frac{i}{2}x^2\cot\alpha-ix(u-\gamma)\operatorname{cosec}\alpha}.\int_{-\infty}^{\infty} e^{\frac{i}{2}t^2\cot\alpha-it(s-\eta)\operatorname{cosec}\alpha} f'(t,x)dt\,dx$$
$$= c_{1\alpha} e^{i(s\tau+u\zeta)} e^{\frac{i}{2}[(s-\eta)^2+(u-\gamma)^2\cot\alpha]}\int_{-\infty}^{\infty} e^{\frac{i}{2}x^2\cot\alpha-ix(u-\gamma)\operatorname{cosec}\alpha}\left\{\left[e^{\frac{i}{2}t^2\cot\alpha-it(s-\eta)\operatorname{cosec}\alpha} f(t,x)\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} e^{\frac{i}{2}t^2\cot\alpha-it(s-\eta)\operatorname{cosec}\alpha}\left(\frac{i}{2}2t\cot\alpha - - i(s-\eta)\operatorname{cosec}\alpha\right) f(t,x)dt\right\}dx$$
$$= c_{1\alpha} e^{i(s\tau+u\zeta)} e^{\frac{i}{2}[(s-\eta)^2+(u-\gamma)^2\cot\alpha]}\int_{-\infty}^{\infty} e^{\frac{i}{2}x^2\cot\alpha-ix(u-\gamma)\operatorname{cosec}\alpha}\left\{-\int_{-\infty}^{\infty} e^{\frac{i}{2}t^2\cot\alpha-it(s-\eta)\operatorname{cosec}\alpha}(it\cot\alpha - i(s-\eta)\operatorname{cosec}\alpha f(t,x)dt)\right\}dy$$
$$= i\cot\alpha\, c_{1\alpha} e^{i(s\tau+u\zeta)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2}[((s-\eta)^2+(u-\gamma)^2)\cot\alpha]}.\,e^{\frac{i}{2}[(t^2+x^2)\cot\alpha-i(t(s-\eta)+x(u-\gamma))]\operatorname{cosec}\alpha} f(t,x)dt\,dx$$
$$+ i(s-\eta)\operatorname{cosec}\alpha\, c_{1\alpha} e^{i(s\tau+u\zeta)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2}[((s-\eta)^2+(u-\gamma)^2)\cot\alpha]}.\,e^{\frac{i}{2}[(t^2+x^2)\cot\alpha-i(t(s-\eta)+x(u-\gamma))]\operatorname{cosec}\alpha} f(t,x)dt\,dx$$

$$= (-i\cot\alpha)\text{ 2D Offst FRFT}\{t f(t,x)\}(s-\eta,u-\gamma) + (i(s-\eta)\operatorname{cosec}\alpha)\text{ 2D Offset FRFT}\{f(t,x)\}(s-\eta,u-\gamma)$$

Similarly we can find

$$(-i\cot\alpha)\text{ 2D Offset FRFT}\{xf(t,x)\}(s-\eta,u-\gamma) + (i(u-\gamma)\operatorname{cosec}\alpha)\text{ 2D Offset FRFT}\{f(t,x)\}(s-\eta,u-\gamma)$$

C. 1st shifting property

$$\text{2D Offset FRFT}\left\{e^{i(at+bx)} f(t,x)\right\}(s-\eta,u-\gamma)$$
$$= e^{\left(\frac{a^2+b^2}{2}\right)\sin 2\alpha} e^{i[a(s-\eta-a\sin\alpha)+b(u-\gamma-b\sin\alpha)]\cos\alpha}$$
$$\text{2D Offset FRFT}\{f(t,x)\}(s-\eta-a\sin\alpha, u-\gamma-b\sin\alpha)$$

Consider

$$2\text{D Offset FRFT}\{e^{i(at+bx)}f(t,x)\}(s-\eta,u-\gamma)$$

$$= c_{1\alpha}\, e^{i(s\tau+u\zeta)}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{1}{2\sin\alpha}[t^2+x^2+(s-\eta)^2+(u-\gamma)^2]\cos\alpha-2[t(s-\eta)+x(u-\gamma)]\operatorname{cosec}\alpha}$$

$$e^{i(at+bx)}f(t,x)\,dt\,dx$$

$$= c_{1\alpha}\, e^{i(s\tau+u\zeta)} e^{\frac{i}{2}[(s-\eta)^2+(u-\gamma)^2]\cot\alpha}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2}\left[(t^2+x^2)\cot\alpha-i[t(s-\eta)+x(u-\gamma)]\right]\operatorname{cosec}\alpha} e^{i(at+bx)}$$

$$f(t,x)\,dt\,dx$$

$$= c_{1\alpha} e^{i(s\tau+u\zeta)} e^{\frac{i}{2}[(s-\eta)^2+(u-\gamma)^2]\cot\alpha}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2}\left[((s-\eta)^2+(u-\gamma)^2)\cot\alpha-it[(s-\eta)\operatorname{cosec}\alpha-a\sin\alpha\operatorname{cosec}\alpha]-ix[(u-\gamma)\operatorname{cosec}\alpha-b\sin\alpha\operatorname{cosec}\alpha]\right]} f(t,x)\,dt\,dx$$

$$= c_{1\alpha} e^{i(s\tau+u\zeta)}\ e^{\frac{i}{2}[(s-\eta)^2+(u-\gamma)^2]\cot\alpha}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2}\left[(t^2+x^2)\cot\alpha-it\operatorname{cosec}\alpha[(s-\eta)-a\sin\alpha]-ix\operatorname{cosec}\alpha[(u-\gamma)-b\sin\alpha]\right]}$$

$$f(t,x)\,dt\,dx$$

$$= c_{1\alpha}\, e^{i(s\tau+u\zeta)} e^{\frac{i}{2}\left[((s-\eta)+a\sin\alpha-a\sin\alpha)^2+((u-\gamma)+b\sin\alpha-b\sin\alpha)^2\right]\cot\alpha}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2}\left[(t^2+x^2)\cot\alpha-it\operatorname{cosec}\alpha[(s-\eta)-a\sin\alpha]-ix\operatorname{cosec}\alpha[(u-\gamma)-b\sin\alpha]\right]}$$

$$f(t,x)\,dt\,dx$$

$$= c_{1\alpha}\, e^{i(s\tau+u\zeta)}$$

$$e^{\frac{i}{2}\left[((s-\eta)-a\sin\alpha)^2-2a\sin\alpha(-(s-\eta)+a\sin\alpha)+\left(((u-\gamma)-b\sin\alpha)^2-2b\sin\alpha(-(u-\gamma)+b\sin\alpha)\right)\right]\cot\alpha}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2}\left[(t^2+x^2)\cot\alpha-it\operatorname{cosec}\alpha[(s-\eta)-a\sin\alpha]-ix\operatorname{cosec}\alpha[(u-\gamma)-b\sin\alpha]\right]}$$

$$e^{(a^2+b^2)\sin^2\alpha\cos\alpha} f(t,x)\,dt\,dx$$

$$= c_{1\alpha}\, e^{i(s\tau+u\zeta)} e^{\frac{i}{2}[(2a(s-\eta)-a\sin\alpha)\cos\alpha]} e^{\frac{i}{2}[(2b(u-\gamma)-b\sin\alpha)\cos\alpha]}$$

$$e^{(a^2+b^2)\sin\alpha\cos\alpha}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c_{1\alpha} e^{\frac{i}{2}\left[((s-\eta)-a\sin\alpha)^2+((u-\gamma)-b\sin\alpha)^2\right]}$$

$$e^{\frac{i}{2}\left[(t^2+x^2)\cot\alpha-it\operatorname{cosec}\alpha[(s-\eta)-a\sin\alpha]-ix\operatorname{cosec}\alpha[(u-\gamma)-b\sin\alpha]\right]}$$

$$f(t,x)\,dt\,dx$$

$$= e^{\left(\frac{a^2+b^2}{2}\right)\sin 2\alpha} e^{i[a((s-\eta)-\sin\alpha)+b((u-\gamma)-\sin\alpha)]\cos\alpha}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c_{1\alpha}\, e^{i(s\tau+u\zeta)}$$

$$e^{\frac{i}{2\sin\alpha}\left[t^2+x^2+((s-\eta)-a\sin\alpha)^2+((u-\gamma)-b\sin\alpha)^2\right]\cos\alpha-2[t[(s-\eta)-a\sin\alpha]+x[(u-\gamma)-b\sin\alpha]]}$$

$$f(t,x)\,dt\,dx$$

$$= e^{\left(\frac{a^2+b^2}{2}\right)\sin 2\alpha} e^{i[a((s-\eta)-\sin\alpha)+b((u-\gamma)-\sin\alpha)]\cos\alpha}$$

$$[2\text{D Offset FRFT}(f(t,x)]\big((s-\eta)-a\sin\alpha\big)[(u-\gamma)-b\sin\alpha]$$

D. Scaling property

$$2\text{D Offset FRFT}\{f(at,bx)\}(s-\eta,u-\gamma) = \frac{1}{ab}\{2\text{D Offset FRFT } g(t,x)\}((s-\eta)(u-\gamma))$$

Where

$$g(t,x) = e^{\frac{i}{2}}\left\{\left(\frac{1-a}{a}\right)\left[\left(\frac{1+a}{a}\right)(at)^2\cot\alpha - 2(s-\eta)at\operatorname{cosec}\alpha\right] + \left(\frac{1-b}{b}\right)\left[\left(\frac{1+b}{b}\right)(bx)^2\cot\alpha - 2(u-\gamma)bx\operatorname{cosec}\alpha\right]\right\}$$

Consider

$$2\text{D Offset FRFT}\{f(at,bx)\}(s-\eta,u-\gamma)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c_{1\alpha}\ e^{i(s\tau+u\zeta)}$$

$$e^{\frac{i}{2\sin\alpha}[t^2+x^2+(s-\eta)^2+(u-\gamma)^2]\cos\alpha-2[t(s-\eta)+x(u-\gamma)]}$$

$$f(at,bx)\,dt\,dx$$

$$= c_{1\alpha} e^{i(s\tau+u\zeta)} e^{\frac{i}{2}[(s-\eta)^2+(u-\gamma)^2]\cot\alpha}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2}[t^2+x^2]\cot\alpha-i[t(s-\eta)+x(u-\gamma)]\operatorname{cosec}\alpha} f(at,bx)\,dt\,dx$$

Putting

$$at = P, \qquad bx = Q$$

$$a\,dt = dP, \qquad b\,dx = dQ$$

$$dt = \frac{dP}{a}, \qquad dx = \frac{dQ}{b}$$

$$2\text{D Offset FRFT}\{f(at,bx)\}(s-\eta,u-\gamma)$$

$$= c_{1\alpha} e^{i(s\tau+u\zeta)} e^{\frac{i}{2}[(s-\eta)^2+(u-\gamma)^2]\cot\alpha}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2}\left[\frac{P^2}{a^2}+\frac{Q^2}{b^2}\right]\cot\alpha-i\left[\frac{P}{a}(s-\eta)+\frac{Q}{b}(u-\gamma)\right]\operatorname{cosec}\alpha} f(P,Q)\frac{dP}{a}\frac{dQ}{b}$$

$$= c_{1\alpha} e^{i(s\tau+u\zeta)} e^{\frac{i}{2}[(s-\eta)^2+(u-\gamma)^2]\cot\alpha}\frac{1}{ab}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2}\left[1+\frac{1-a^2}{a^2}\right]P^2 \cot\alpha}\, e^{\frac{i}{2}\left[1+\frac{1-b^2}{b^2}\right]Q^2 \cot\alpha}$$

$$e^{-i\left[\frac{P}{a}(s-\eta)+\frac{Q}{b}(u-\gamma)\right]\operatorname{cosec}\alpha} f(P,Q)\,dP\,dQ$$

$$= \frac{c_{1\alpha}}{ab} e^{i(s\tau+u\zeta)} e^{\frac{i}{2}[(s-\eta)^2+(u-\gamma)^2]\cot\alpha}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2}[P^2+Q^2]\cot\alpha - i[(s-\eta)P+(u-\gamma)Q]\operatorname{cosec}\alpha}$$

$$e^{\frac{i}{2}\left[\left(\frac{1-a^2}{a^2}\right)P^2+\left(\frac{1-b^2}{b^2}\right)Q^2\right]\cot\alpha - i\left[(s-\eta)P\left(\frac{1-a}{a}\right)+(u-\gamma)Q\left(\frac{1-b}{b}\right)\right]\operatorname{cosec}\alpha}$$

$$f(P,Q)\,dP\,dQ$$

$$= \frac{1}{ab}\{\text{2D Offset FRFT}$$

$$e^{\frac{i}{2}\left[\left(\frac{1-a^2}{a^2}\right)P^2+\left(\frac{1-b^2}{b^2}\right)Q^2\right]\cot\alpha - i\left[(s-\eta)P\left(\frac{1-a}{a}\right)+(u-\gamma)Q\left(\frac{1-b}{b}\right)\right]\operatorname{cosec}\alpha}$$

$$f(P,Q)(s-\eta, u-\gamma)\}$$

$$= \frac{1}{ab}\{[\text{2D Offset FRFT } g(t,x)](s-\eta, u-\gamma)\}$$

Where

$$g(t,x)$$

$$= e^{\frac{i}{2}\left[\left(\frac{1-a^2}{a^2}\right)P^2+\left(\frac{1-b^2}{b^2}\right)Q^2\right]\cot\alpha - i\left[(s-\eta)P\left(\frac{1-a}{a}\right)+(u-\gamma)Q\left(\frac{1-b}{b}\right)\right]\operatorname{cosec}\alpha}$$

$$f(P,Q)$$

$$= e^{\frac{i}{2}\left[\left(\frac{(1-a)(1+a)}{a^2}\right)P^2+\left(\frac{(1-b)(1+b)}{b^2}\right)Q^2\right]\cot\alpha - i\left[(s-\eta)P\left(\frac{1-a}{a}\right)+(u-\gamma)Q\left(\frac{1-b}{b}\right)\right]\operatorname{cosec}\alpha}$$

$$f(P,Q)$$

$$= e^{\frac{i}{2}\left[\left(\frac{1-a}{a}\right)\left[\frac{1+a}{a}P^2\cot\alpha - 2(s-\eta)P\operatorname{cosec}\alpha\right]\right]}.$$

$$e^{\frac{i}{2}\left[\left(\frac{1-b}{b}\right)\left[\frac{1+b}{b}Q^2\cot\alpha - 2(u-\gamma)Q\operatorname{cosec}\alpha\right]\right]} f(P,Q)$$

$$= e^{\frac{i}{2}\left[\left(\frac{1-a}{a}\right)\left[\frac{1+a}{a}(at)^2\cot\alpha - 2(s-\eta)(at)\operatorname{cosec}\alpha\right]\right]}.$$

$$e^{\frac{i}{2}\left[\left(\frac{1-b}{b}\right)\left[\frac{1+b}{b}(bx)^2\cot\alpha - 2(u-\gamma)(bx)\operatorname{cosec}\alpha\right]\right]} f(P,Q)$$

VI. CONCLUSION

In the present work generalization of Two Dimensional Offset Fractional Fourier Transform is presented. And operation transform formulae are proved. The Offset transform and their generalization can be useful in optical system to investigate self-imaging phenomenon that is when the output is a (possibly scaled) duplicate of the original.

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