

INVARIANT SUBMANIFOLD OF $(3k, k)$ STRUCTURE MANIFOLD

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ABSTRACT

In this paper, we have studied various properties of a $(3k, k)$ structure manifold and its invariant submanifold, where k is positive integer. Under two different assumptions, the nature of induced structure ψ , has also been discussed.

Keywords : Invariant submanifold, Nijenhuis tensor, projection operators and complementary distributions.

1 Introduction

Let V^m be a C^∞ m -dimensional Riemannian manifold imbedded in a C^∞ n -dimensional Riemannian manifold M^n , where $m < n$. The imbedding being denoted by

$$f : V^m \longrightarrow M^n$$

Let B be the mapping induced by f i.e. $B = df$

$$df : T(V) \longrightarrow T(M)$$

Let $T(V, M)$ be the set of all vectors tangent to the submanifold $f(V)$. It is well known that

$$B : T(V) \longrightarrow T(V, M)$$

is an isomorphism. The set of all vectors normal to $f(V)$ forms a vector bundle over $f(V)$, which we shall denote by $N(V, M)$. We call $N(V, M)$ the normal bundle of V^m . The vector bundle induced by f from $N(V, M)$ is denoted by $N(V)$. We denote

by $C : N(V) \longrightarrow N(V, M)$ the natural isomorphism and by $\eta_s^r(V)$ the space of all C^∞ tensor fields of type (r, s) associated with $N(V)$. Thus $\zeta_0^0(V) = \eta_0^0(V)$ is the space of all C^∞ functions defined on V^m while an element of $\eta_0^1(V)$ is a C^∞ vector field normal to V^m and an element of $\zeta_0^1(V)$ is a C^∞ vector field tangential to V^m .

Let \bar{X} and \bar{Y} be vector fields defined along $f(V)$ and \tilde{X}, \tilde{Y} be the local extensions of \bar{X} and \bar{Y} respectively. Then $[\tilde{X}, \tilde{Y}]$ is a vector field tangential to M^n and its restriction $[\tilde{X}, \tilde{Y}]/f(V)$ to $f(V)$ is determined independently of the choice of these local extension \tilde{X} and \tilde{Y} . Thus $[\bar{X}, \bar{Y}]$ is defined as

$$(1.1) \quad [\bar{X}, \bar{Y}] = [\tilde{X}, \tilde{Y}]/f(V)$$

Since B is an isomorphism

$$(1.2) \quad [BX, BY] = B[X, Y] \quad \text{for all } X, Y \in \zeta_0^1(V)$$

Let \bar{G} be the Riemannian metric tensor of M^n , we define g and g^* on V^m and $N(V)$ respectively as

$$(1.3) \quad g(X_1, X_2) = \tilde{G}(BX_1, BX_2) f, \quad \text{and}$$

$$(1.4) \quad g^*(N_1, N_2) = \tilde{G}(CN_1, CN_2)$$

For all $X_1, X_2 \in \zeta_0^1(V)$ and $N_1, N_2 \in \eta_0^1(V)$

It can be verified that g and g^* are the induced metrics on V^m and $N(V)$ respectively.

Let us suppose that M^n is a $(2k + S, S)$ structure manifold with structure tensor $\tilde{\psi}$ of type (1,1) satisfying

$$(1.5) \quad \tilde{\psi}^{3k} + \tilde{\psi}^k = 0$$

Let \tilde{L} and \tilde{M} be the complementary distributions corresponding to the projection operators

$$(1.6) \quad \tilde{l} = -\tilde{\psi}^{2k}, \quad \tilde{m} = I + \tilde{\psi}^{2k}$$

where I denotes the identity operator.

From (1.5) and (1.6), we have

$$(1.7) \quad \text{(a)} \quad \tilde{l} + \tilde{m} = I \quad \text{(b)} \quad \tilde{l}^2 = \tilde{l} \quad \text{(c)} \quad \tilde{m}^2 = \tilde{m}$$

$$(d) \quad \tilde{l} \tilde{m} = \tilde{m} \tilde{l} = 0$$

Let D_l and D_m be the subspaces inherited by complementary projection operators l and m respectively.

We define

$$D_l = \{X \in T_p(V) : lX = X, mX = 0\}$$

$$D_m = \{X \in T_p(V) : mX = X, lX = 0\}$$

Thus $T_p(V) = D_l + D_m$

$$\text{Also } Ker l = \{X : lX = 0\} = D_m$$

$$Ker m = \{X : mX = 0\} = D_l$$

at each point p of $f(V)$.

2. INVARIANT SUBMANIFOLD OF $(3k, k)$ STRUCTURE MANIFOLD

We call V^m to be invariant submanifold of M^n if the tangent space $T^p(f(V))$ of $f(V)$ is invariant by the linear mapping $\tilde{\psi}$ at each point p of $f(V)$. Thus

$$(2.1) \quad \tilde{\psi}BX = B\psi X, \text{ for all } X \in \zeta_0^1(V), \text{ and } \psi \text{ being a } (1,1) \text{ tensor field in } V^m.$$

Theorem (2.1): Let \tilde{N} and N be the Nijenhuis tensors determined by $\tilde{\psi}$ and ψ in M^n and V^m respectively, then

$$(2.2) \quad \tilde{N}(BX, BY) = BN(X, Y), \text{ for all } X, Y \in \zeta_0^1(V)$$

Proof : We have, by using (1.2) and (2.1)

$$(2.3) \quad \begin{aligned} \tilde{N}(BX, BY) &= [\tilde{\psi}BX, \tilde{\psi}BY] + \tilde{\psi}^2[BX, BY] \\ &\quad - \tilde{\psi}[\tilde{\psi}BX, BY] - \tilde{\psi}[BX, \tilde{\psi}BY] \end{aligned}$$

Simplifying the expression, we get (2.2),

3. DISTRIBUTION \tilde{M} NEVER BEING TANGENTIAL TO $f(V)$

Theorem (3.1) if the distribution \tilde{M} is never tangential to $f(V)$, then

$$(3.1) \quad \tilde{m}(BX) = 0 \quad \text{for all} \quad X \in \zeta_0^1(V)$$

and the induced structure ψ on V^m satisfies

$$(3.2) \quad \psi^{2k} = -I$$

Proof: if possible $\tilde{m}(BX) \neq 0$. From (2.1) We get

$$(3.3) \quad \tilde{\psi}^{2k}BX = B\psi^{2k}X; \text{ from (1.6) and (3.3)}$$

$$\begin{aligned} \tilde{m}(BX) &= (I + \tilde{\psi}^{2k})BX \\ &= BX + B\psi^{2k}X \end{aligned}$$

$$(3.4) \quad \tilde{m}(BX) = B[X + \psi^{2k}X]$$

This relation shows that $\tilde{m}(BX)$ is tangential to $f(V)$ which contradicts the hypothesis. Thus $\tilde{m}(BX) = 0$. Using this result in (3.4) and remembering that B is an isomorphism, We get

$$(3.5) \quad \psi^{2k} = -I$$

Theorem (3.2) Let \tilde{M} be never tangential to $f(V)$, then

$$(3.6) \quad \tilde{N}_{\tilde{m}}(BX, BY) = 0$$

Proof: We have

$$(3.7) \quad \begin{aligned} \tilde{N}_{\tilde{m}}(BX, BY) &= [\tilde{m}BX, \tilde{m}BY] + \tilde{m}^2[BX, BY] \\ &\quad - \tilde{m}[\tilde{m}BX, BY] - \tilde{m}[BX, \tilde{m}BY] \end{aligned}$$

Using (1.2), (1.7) (c) and (3.1), we get (3.6).

Theorem (3.3) Let \tilde{M} be never tangential to $f(V)$, then

$$(3.8) \quad \tilde{N}_i(BX, BY) = 0$$

Proof: We have

$$(3.9) \quad \begin{aligned} \tilde{N}_i(BX, BY) &= [\tilde{l}BX, \tilde{l}BY] + \tilde{l}^2[BX, BY] - \tilde{l}[\tilde{l}BX, BY] \\ &\quad - \tilde{l}[BX, \tilde{l}BY] \end{aligned}$$

Using (1.2), (1.7) (a), (b) and (3.1) in (3.9); we get (3.8)

Theorem (3.4) Let \tilde{M} be never tangential to $f(V)$. Define

$$(3.10) \quad \begin{aligned} \tilde{H}(\tilde{X}, \tilde{Y}) &= \tilde{N}(\tilde{X}, \tilde{Y}) - \tilde{N}(\tilde{m}\tilde{X}, \tilde{Y}) - \tilde{N}(\tilde{X}, \tilde{m}\tilde{Y}) \\ &\quad + \tilde{N}(\tilde{m}\tilde{X}, \tilde{m}\tilde{Y}) \end{aligned}$$

For all $\tilde{X}, \tilde{Y} \in \zeta_0^1(M)$, then

$$(3.11) \quad \tilde{H}(BX, BY) = BN(X, Y)$$

Proof: Using $\tilde{X} = BX$, $\tilde{Y} = BY$ and (2.2), (3.1) in (3.10) We get (3.11).

4. **DISTRIBUTION \tilde{M} ALWAYS BEING TANGENTIAL TO $f(V)$**

Theorem (4.1) Let \tilde{M} be always tangential to $f(V)$, then

$$(4.1) \text{ (a) } \tilde{m}(BX) = Bm X \quad \text{(b) } \tilde{l}(BX) = Bl X$$

Proof: from (3.4), We get (4.1) (a). Also

$$(4.2) \quad l = -\psi^{2k}$$

$$lX = -\psi^{2k} X$$

$$(4.3) \quad BlX = -B\psi^{2k} X$$

Using (2.1) in (4.3)

$$(4.4) \quad BlX = -\tilde{\psi}^{2k} BX = \tilde{l}(BX),$$

which is (4.1) (b).

Theorem (4.2) Let \tilde{M} be always tangential to $f(V)$, then l and m satisfy

$$(4.5) \text{ (a) } l+m=l \text{ (b) } lm=ml=0 \text{ (c) } l^2=l \text{ (d) } m^2=m.$$

Proof: Using (1.7) and (4.1) We get the results.

Theorem (4.3) If \tilde{M} is always tangential to $f(V)$, then

$$(4.6) \quad \psi^{3k} + \psi^k = 0$$

Proof: From (2.1)

$$(4.7) \quad \tilde{\psi}^{3k} BX = B\psi^{3k} X$$

Using (1.5) in (4.7)

$$-\tilde{\psi}^k BX = B\psi^{3k} X$$

$$-B\psi^k X = B\psi^{3k} X$$

Or $\psi^{3k} + \psi^k = 0$ which is (4.6)

Theorem (4.4) : If \tilde{M} Is always tangential to $f(V)$ then as in (3.10)

$$(4.8) \quad \tilde{H}(BX, BY) = BH(X, Y)$$

Proof: from (3.10) we get

$$(4.9) \quad \tilde{H}(BX, BY) = \tilde{N}(BX, BY) - \tilde{N}(\tilde{m}BX, BY) \\ - \tilde{N}(BX, \tilde{m}BY) + \tilde{N}(\tilde{m}BX, \tilde{m}BY)$$

Using (4.1) (a) and (2.2) in (4.9) we get (4.8).

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