A COMMON FIXED POINT THEOREM SATISFYING CONTRACTIVE CONDITION OF INTEGRAL TYPE

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ABSTRACT

Fixed point techniques have been applied in such diverse fields as economics, engineering, game theory, and physics. The aim of this paper is to report new fixed point theorem. In this paper we prove a common fixed point theorem. The existence of fixed point for weakly compatible maps is proved under contractive condition of integral type.

Keywords: *Fixed point, complete metric space, Weakly Compatible maps.*

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1. INTRODUCTION

Commutativity is a widely used topic in the field of Mathematics and engineering, especially in the field of pure mathematics. In fixed point theory it is mostly used by many authors. Very first the concept of commutativity was introduced by Jungck [7] which generalizing the Banach's fixed point theorem and after that it has been used and generalized by many authors in several ways. In the continuing process, Sessa [14] has introduced the concept of weakly commuting. This result was further generalized by Jungck [8], so called compatibility. An Important result which is verifies easily that when two mappings are commuting then they are compatible. Clearly commuting, weakly commuting mappings are compatible but conversely need not be true.

The study of fixed point theorems satisfying various types of contractive inequalities has been a very active field of research during the last few decades. Such condition involves rational, irrational and general type expressions. To study more about this matter we recommended going deep into the survey articles by Rhodes [10], [11].

In [3] Branciari obtained a fixed point result for a single mapping satisfying an analogue of a Banach contraction principle for integral type inequality as below:

Theorem 1.1[3] Let (X, d) be a complete metric space, $\beta \in [0,1), f: X \to X$ a mapping such that for each $x, y \in X$,

$$\int_0^{d(fx,fy)} \varphi(t)dt \leq \beta \int_0^{d(x,y)} \varphi(t)dt$$

Where $\varphi: [0,+\infty) \to [0,+\infty)$ is a "Lebesgue-integrable function" which is summable on each compact subset of R^+, non-negative, and such that for each $\epsilon > 0$, $\int_0^{\epsilon} \varphi(t)dt > 0$. then P has a unique fixed point such that for each $x \in X$, $\lim_{n\to\infty} f^n x = z$.

After this result, there are many theorems dealing with mappings satisfying a general contractive condition of integral type.

R.K Saini, Vishal Gupta[5] prove a common fixed point theorem for R-weakly commuting fuzzy maps satisfying a general contractive condition of integral type. Recently Vishal Gupta [6] proves a Common Fixed Point Theorem for Compatible Mapping.

2. PRELIMINARIES

We recall the definitions of complete metric space and other results that will be needed in the sequel.

Definition 2.1. Let f and g be two self maps on a set X. Maps f and g are said to be commuting if $fgx = gfx$ for all $x \in X$.

Definition 2.2. Let f and g be two self maps on a set X. If $fx = gx$, for some $x \in X$ then

x *is called coincidence point of* f *and* g.

Definition 2.3. *Let* f *and* g *be two selfmaps on a set* X, *then* f *and* g *are said to be weakly compatiable if they commute at their coincidence points.*

Lemma 2.4. *Let* f *and* g *be weakly compatiable self mapping of a set* X. *If* f *and* g *have a unique point of coincidence* ω, *then* ω *is the unique common fixed of* f *and* g.

Definition 2.5. *A sequence* $\{x_n\}$ *in a metric space* (X, d) *is said to be convergent to a point* $x \in X$, *if* $\lim_{n\to\infty} d(x_n, x) = 0$

Definition 2.6. *A sequence* $\{x_n\}$ *in a metric space* (X, d) *is said to be cauchy sequence if* $\lim_{t\to\infty} d(x_n, x_m) = 0$ *for all* $n, m > t$.

Definition 2.7. *A metric space* (X, d) *is said to be complete if every cauchy sequence in* X *is convergent.*

3. MAIN RESULT

Theorem3.1. Let S and T be self compatible maps of a complete metric space (X, d) satisfying the following conditions

(i) $S(X) \subset T(X)$ (3.1)

(ii) $\int_0^{d(Sx,Sy)} \varphi(t)dt \le \int_0^{d(Tx,Ty)} \varphi(t)dt - \phi \int_0^{d(Tx,Ty)} \varphi(t)dt$ (3.2)

for each x,y ϵ X where $\phi: [0, +\infty) \to [0, +\infty)$ is a lower semi continuous and non decreasing function such that $\phi(t) = 0$ if and only if $t = 0$ also $\varphi: [0, +\infty) \to [0, +\infty)$ is a "Lebesgue-integrable function" which is summable on each compact subset of R^+, non-negative, and such that for each $\epsilon > 0$, $\int_0^{\epsilon} \varphi(t)dt > 0$. Then S and T have a unique common fixed point.

Proof: Let x_0 be an arbitrary point of X. Since S(X) ⊂ T(X).Choose a point $x_1$in X such that $Sx_0=Tx_1$. Continuing this process, in general, choose x_{n+1} such that $y_n = Tx_{n+1} = Sx_n$, $n = 0,1,2 \ldots\ldots$

For each integer $n \ge 1$,from (3.2)

$$\int_0^{d(y_n,y_{n+1})} \varphi(t)dt \le \int_0^{d(y_{n-1},y_n)} \varphi(t)dt - \phi \int_0^{d(y_{n-1},y_n)} \varphi(t)dt \quad (3.3)$$

$$\le \int_0^{d(y_{n-1},y_n)} \varphi(t)dt$$

Thus

$$\int_0^{d(y_n,y_{n+1})} \varphi(t)dt \le \int_0^{d(y_{n-1},y_n)} \varphi(t)dt$$

Let us take $Z_n = \int_0^{d(y_n, y_{n+1})} \varphi(t)dt$, then it follows that Z_n is monotone decreasing and lower bounded sequence of numbers. Therefore there exist $k \geq 0$ such that $Z_n \to k$ as $n \to \infty$. Suppose that $k > 0$.

Taking limit as $n \to \infty$ on both sides of (3.3) and using that ϕ is lower semi continuous, we get,

$$k \leq k - \phi(k) < k$$

This is a contradiction. Therefore $k = 0$. This implies

$$Z_n \to 0 \text{ as } n \to \infty$$

$$\int_0^{d(y_n, y_{n+1})} \varphi(t)dt \to 0 \text{ as } n \to \infty \quad (3.5)$$

Now we prove that $\{y_n\}$ is a Cauchy sequence. Suppose it is not. Therefore there exist an $\varepsilon > 0$ and subsequence $\{y_{m(p)}\}$ & $\{y_{n(p)}\}$ such that for each positive integer p, $n(p)$is minimal in the sense that

$$d(y_{n(p)}, y_{m(p)}) \geq \varepsilon, \quad d(y_{n(p)-1}, y_{m(p)}) < \varepsilon \quad (3.6)$$

Now,

$$\varepsilon \leq d(y_{n(p)}, y_{m(p)}) \leq d(y_{n(p)}, y_{n(p)-1}) + d(y_{n(p)-1}, y_{m(p)})$$
$$< \varepsilon + d(y_{n(p)}, y_{n(p)-1}) \quad (3.7)$$

Now

$$0 < \mathcal{L} = \int_0^{\varepsilon} \varphi(t)dt \leq \int_0^{d(y_{n(p)}, y_{m(p)})} \varphi(t)dt \leq \int_0^{\varepsilon + d(y_{n(p)}, y_{n(p)-1})} \varphi(t)dt$$

Letting $p \to \infty$ and from (3.5)

$$\lim_{p \to \infty} \int_0^{d(y_{n(p)}, y_{m(p)})} \varphi(t)dt = \mathcal{L} \quad (3.8)$$

Now consider the triangle inequality,

$$d(y_{n(p)}, y_{m(p)}) \leq d(y_{n(p)}, y_{n(p)-1}) + d(y_{n(p)-1}, y_{m(p)-1}) + d(y_{m(p)-1}, y_{m(p)})$$
$$d(y_{n(p)-1}, y_{m(p)-1}) \leq d(y_{n(p)-1}, y_{n(p)}) + d(y_{n(p)}, y_{m(p)}) + d(y_{m(p)}, y_{m(p)-1})$$

and therefore,

$$\int_0^{d(y_{n(p)}, y_{m(p)})} \varphi(t)dt \leq \int_0^{d(y_{n(p)}, y_{n(p)-1}) + d(y_{n(p)-1}, y_{m(p)-1}) + d(y_{m(p)-1}, y_{m(p)})} \varphi(t)dt$$
$$\int_0^{d(y_{n(p)-1}, y_{m(p)-1})} \varphi(t)dt \leq \int_0^{d(y_{n(p)-1}, y_{n(p)}) + d(y_{n(p)}, y_{m(p)}) + d(y_{m(p)}, y_{m(p)-1})} \varphi(t)dt$$

Taking $p \to \infty$ and using (3.5) and (3.8) in above inequalities, we get

$$\int_0^{d(y_{n(p)-1}, y_{m(p)-1})} \varphi(t)dt \leq \mathcal{L} \leq \int_0^{d(y_{n(p)-1}, y_{m(p)-1})} \varphi(t)dt$$

This implies,

$$\lim_{p \to \infty} \int_0^{d(y_{n(p)-1}, y_{m(p)-1})} \varphi(t)dt = \mathcal{L} \quad (3.9)$$

Now from (3.2), we have

$$\int_0^{d(y_{n(p)}, y_{m(p)})} \varphi(t)dt \leq \int_0^{d(y_{n(p)-1}, y_{m(p)-1})} \varphi(t)dt - \phi \int_0^{d(y_{n(p)-1}, y_{m(p)-1})} \varphi(t)dt$$

Taking limit as $p \to \infty$ and using (3.8) and (3.9), we get

$$(\mathcal{L}) \leq (\mathcal{L}) - \phi(\mathcal{L})$$

$$(\mathcal{L}) < (\mathcal{L})$$

This is a contradiction. Hence $\{y_n\}$ is a Cauchy sequence. Since (X, d) is complete metric space, therefore there exist a point $\boldsymbol{v}$ such that

$$Sx_n \to \boldsymbol{v} \;\&\; Tx_n \to \boldsymbol{v} \; as \; n \to \infty$$

Consequently, we can find h in X such that $T(\mathrm{h}) = \boldsymbol{v}$.

Now,

$$\int_0^{d(Sx_n, S\mathrm{h})} \varphi(t)dt \leq \int_0^{d(Tx_n, T\mathrm{h})} \varphi(t)dt - \phi \int_0^{d(Tx_n, T\mathrm{h})} \varphi(t)dt$$

On taking limit as $n \to \infty$ implies, $\psi(\int_0^{d(v, S\mathrm{h})} \varphi(t)dt) = 0$ implies that $S(\mathrm{h}) = \boldsymbol{v}$. Hence $\boldsymbol{v}$ is the point of coincidence of S and T.

Now we prove that $\boldsymbol{v}$ is the unique point of coincidence of S and T. Suppose not, therefore there exist τ($\tau \neq \boldsymbol{v}$) and there exist α in X such that $T(\alpha) = S(\alpha) = \tau$.

Using (3.2) we have

$$\int_0^{d(Tv, T\alpha)} \varphi(t)dt = \int_0^{d(Sv, S\alpha)} \varphi(t)dt$$

$$\leq \int_0^{d(Tv, T\alpha)} \varphi(t)dt - \phi \int_0^{d(Tv, T\alpha)} \varphi(t)dt$$

$$\int_0^{d(Tv, T\alpha)} \varphi(t)dt < \int_0^{d(Tv, T\alpha)} \varphi(t)dt$$

This is a contradiction which implies $\tau = \boldsymbol{v}$. This proves uniqueness of point of coincidence of S and T. Therefore by using lemma (2.4), the result is proved.

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