CONICS: HYPERBOLA

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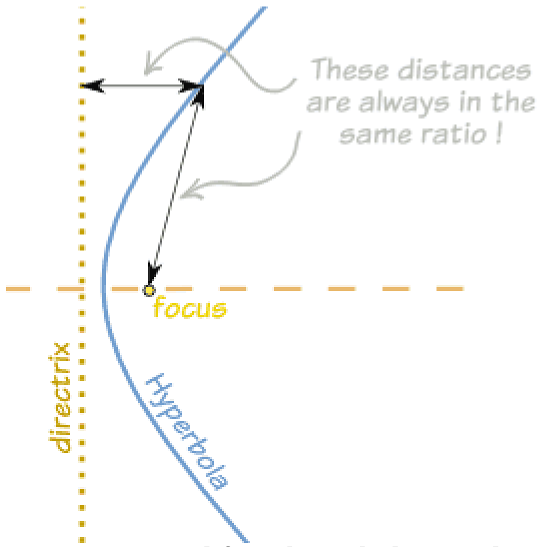
INTRODUCTION

A hyperbola is a curve where the distances of any point from:

- a fixed point (the **focus**), and
- a fixed straight line (the **directrix**) are always in the same ratio.



This ratio is called the eccentricity, and for a hyperbola it is always greater than 1.

The hyperbola is an open curve (has no ends).

A **hyperbola** is a type of smooth curve, lying in a plane, defined by its geometric properties or by equations for which it is the solution set. A hyperbola has two pieces, called connected components or branches that are mirror images of each other and resemble two infinite bows. The hyperbola is one of the four kinds of conic section, formed by the intersection of a plane and a cone. The other conic sections are the parabola, the ellipse, and the circle (the circle is a special case of the ellipse). Which conic section is formed depends on the angle the plane makes with the axis of the cone, compared with the angle a straight line on the surface of the cone makes with the axis of the cone. If the angle between the plane and the axis is less than the angle between the line on the cone and the axis, or if the plane is parallel to the axis, then the plane intersects both halves of the double cone and the conic is a hyperbola.

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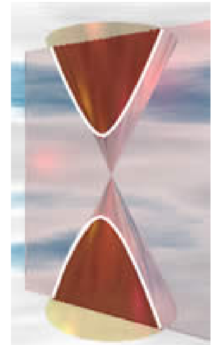
Hyperbolas consist of two vaguely parabola shaped pieces that open either up and down or right and left. Also, just like parabolas each of the pieces has a vertex. Note that they aren't really parabolas, they just resemble parabolas.

There are also two lines on each graph. These lines are called asymptotes and as the graphs show as we make *x* large (in both the positive and negative sense) the graph of the hyperbola gets closer and closer to the asymptotes. The asymptotes are not officially part of the graph of the hyperbola. However, they are usually included so that we can make sure and get the sketch correct. The point where the two asymptotes cross is called the center of the hyperbola.

There are two **standard forms** of the hyperbola, one for each type shown above. Here is a table giving each form as well as the information we can get from each one.

Form	$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$	$\frac{(y-k)^2}{b^2}-\frac{(x-h)^2}{a^2}=1$ $\frac{(y-k)^2}{b^2}-\frac{(x-h)^2}{a^2}=1$
Center	(h,k) (h,k)	(h,k) (h,k)
Opens	Opens left and right	Opens up and down
Vertices	$(h+a,k)$ and $(h-a,k)$ $(h+a,k)$ $(h-a,k)$	$(h,k+b)$ and $(h,k-b)$ $(h,k+b)$ $(h,k-b)$
Slope of Asymptotes	$\pm\frac{b}{a}$ $\pm\frac{b}{a}$	$\pm\frac{b}{a}$ $\pm\frac{b}{a}$
Equations of Asymptotes	$y=k\pm\frac{b}{a}(x-h)$ $y=k\pm\frac{b}{a}(x-h)$	$y=k\pm\frac{b}{a}(x-h)$ $y=k\pm\frac{b}{a}(x-h)$

MATHEMATICAL DEFINITIONS



CONIC SECTION

When you slice through a cone (the slice must be steep - steeper than that for a parabola).

EQUATION

By placing a hyperbola on an x-y graph (centered over the x-axis and y-axis), the equation of the curve is:

$x^2/a^2 - y^2/b^2 = 1$

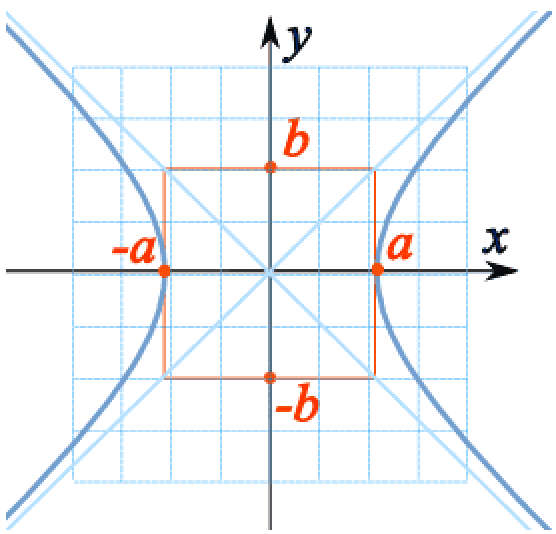
also:

One **vertex** is at (a, 0), and the other is at (-a, 0)

The **asymptotes** are the straight lines:

- $y = (b/a)x$
- $y = -(b/a)x$

And the equation is also similar to the equation of the ellipse: $\mathbf{x^2/a^2 + y^2/b^2 = 1}$, except for a "-" instead of a "+")



ECCENTRICITY

On this diagram:

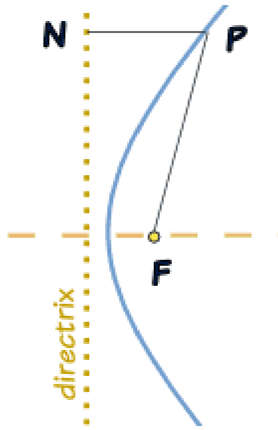
- P is a point on the curve,
- F is the focus and
- N is the point on the directrix so that PN is perpendicular to the directrix.

The ratio PF/PN is the **eccentricity** of the hyperbola (for a hyperbola the eccentricity is always greater than 1).

It can also given by the formula:

$$e = \frac{\sqrt{a^2 + b^2}}{a}$$

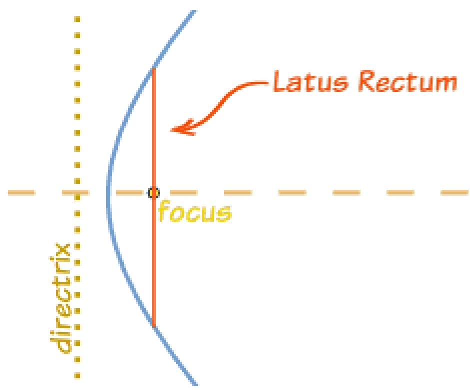
Using "a" and "b" from the diagram above



LATUS RECTUM

The Latus Rectum is the line through the focus and parallel to the directrix.

The length of the Latus Rectum is $2b^2/a$.



HOW ECCENTRICITIES OF THE ELLIPSE AND HYPERBOLA ARE RECIPROCALS.

Let the point P on the hyperbola have Cartesian coordinates (x, y), then the definition of the hyperbola $r_2 - r_1 = 2a$ gives

$$\sqrt{(x-a)^2 + y^2} - \sqrt{(x+a)^2 + y^2} = 2a. \qquad (3)$$

Rearranging and completing the square gives

$$x^2(c^2 - a^2) - a^2 y^2 = a^2(c^2 - a^2), \tag{4}$$

and dividing both sides by $a^2(c^2 - a^2)$ results in

$$\frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} = 1. \tag{5}$$

By analogy with the definition of the ellipse, define

$$b^2 = c^2 - a^2, \tag{6}$$

so the equation for a hyperbola with semimajor axis a parallel to the *x*-axis and semiminor axis b parallel to the *y*-axis is given by

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \tag{7}$$

or, for a center at the point (x_0, y_0) instead of $(0, 0)$,

$$\frac{(x - x_0)^2}{a^2} - \frac{(y - y_0)^2}{b^2} = 1. \tag{8}$$

Unlike the ellipse, no points of the hyperbola actually lie on the semiminor axis, but rather the ratio b/a determines the vertical scaling of the hyperbola. The eccentricity e of the hyperbola (which always satisfies $e > 1$) is then defined as

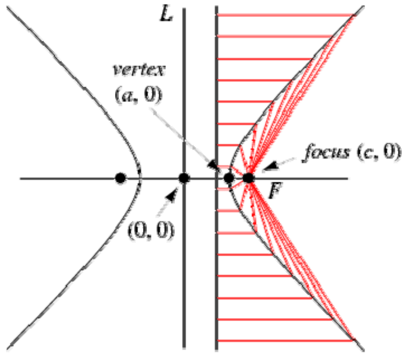
$$e = \frac{c}{a} = \sqrt{1 + \frac{b^2}{a^2}}. \tag{9}$$

In the standard equation of the hyperbola, the center is located at (x_0, y_0), the foci are at $(x_0 \pm c, y_0)$, and the vertices are at $(x_0 \pm a, y_0)$. The so-called asymptotes (shown as the dashed lines in the above figures) can be found by substituting 0 for the 1 on the right side of the general equation (8),

$$y = \pm \frac{b}{a}(x - x_0) + y_0, \tag{10}$$

and therefore have slopes $\pm b/a$.

The special case $a = b$ (the left diagram above) is known as a rectangular hyperbola because the asymptotes are perpendicular.'



The hyperbola can also be defined as the locus of points whose distance from the focus F is proportional to the horizontal distance from a vertical line L known as the conic section directrix, where the ratio is >1. Letting r be the ratio and d the distance from the center at which the directrix lies, then

$$d = \frac{a^2}{c} \quad (11)$$

$$r = \frac{c}{a}, \quad (12)$$

where r is therefore simply the eccentricity e.

Like noncircular ellipses, hyperbolas have *two* distinct foci and two associated conic section directrices, each conic section directrix being perpendicular to the line joining the two foci (Eves 1965, p. 275).

The focal parameter of the hyperbola is

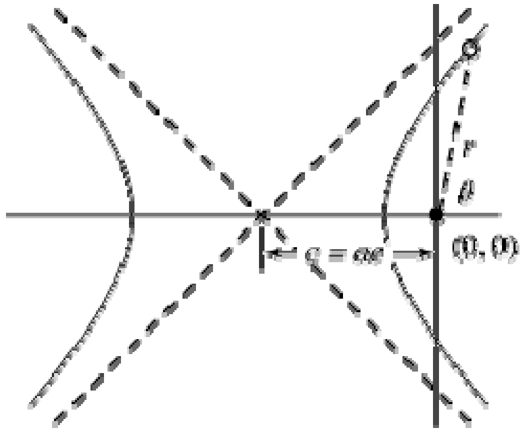
$$p = \frac{b^2}{\sqrt{a^2+b^2}} \quad (13)$$

$$= \frac{c^2-a^2}{c} \quad (14)$$

$$= \frac{a\left(e^2-1\right)}{e}. \quad (15)$$

In polar coordinates, the equation of a hyperbola centered at the origin (i.e., with $x_0 = y_0 = 0$) is

$$r^2 = \frac{a^2 b^2}{b^2 \cos^2\theta - a^2 \sin^2\theta}. \quad (16)$$



In polar coordinates centered at a focus,

$$r = \frac{a\left(e^2 - 1\right)}{1 - e\cos\theta}, \quad (17)$$

as illustrated above.

The two-center bipolar coordinates equation with origin at a focus is

$$r_1 - r_2 = \pm 2a. \quad (18)$$

Parametric equations for the right branch of a hyperbola are given by

$$x = a\cosh t \quad (19)$$

$$y = b\sinh t, \quad (20)$$

where $\cosh x$ is the hyperbolic cosine and $\sinh x$ is the hyperbolic sine, which ranges over the right branch of the hyperbola.

A parametric representation which ranges over both branches of the hyperbola is

$$x = a\sec t \quad (21)$$

$$y = b\tan t, \quad (22)$$

with $t \in (-\pi, \pi)$ and discontinuities at $\pm\pi/2$. The arc length, curvature, and tangential angle for the above parametrization are

$$s(t) = -i\,b\,E\left(i\,t,\ \sqrt{1 + \frac{a^2}{b^2}}\right) \quad (23)$$

$$\kappa(t) = -\frac{a\,b}{\left(b^2\cosh^2 t + a^2\sinh^2 t\right)^{3/2}} \quad (24)$$

$$\phi(t) = -\tan^{-1}\left(\frac{a}{b}\tan t\right), \quad (25)$$

where $E(\phi, k)$ is an elliptic integral of the second kind.

The special affine curvature of the hyperbola is

$$k = -(a\,b)^{-2/3}. \qquad (26)$$

The locus of the apex of a variable cone containing an ellipse fixed in three-space is a hyperbola through the foci of the ellipse. In addition, the locus of the apex of a cone containing that hyperbola is the original ellipse. Furthermore, the eccentricities of the ellipse and hyperbola are reciprocals.

PROPERTIES OF HYPERBOLA:

- If a line intersects one branch of a hyperbola at M and N and intersects the asymptotes at P and Q, then MN has the same midpoint as PQ.
- The following are concurrent: (1) a circle passing through the hyperbola's foci and centered at the hyperbola's center; (2) either of the lines that are tangent to the hyperbola at the vertices; and (3) either of the asymptotes of the hyperbola
- The following are also concurrent: (1) the circle that is centered at the hyperbola's center and that passes through the hyperbola's vertices; (2) either directrix; and (3) either of the asymptotes.
- The area of a triangle two of whose sides lie on the asymptotes, and whose third side is tangent to the hyperbola, is independent of the location of the tangency point. Specifically, the area is ab, where a is the semi-major axis and b is the semi-minor axis.
- The distance from either focus to either asymptote is b, the semi-minor axis; the nearest point to a focus on an asymptote lies at a distance from the center equal to a, the semi-major axis. Then using the Pythagorean theorem on the right triangle with these two segments as legs shows that $a^2 + b^2 = c^2$, where c is the semi-focal length (the distance from a focus to the hyperbola's center).

APPLICATIONS:

Sundials

Hyperbolas may be seen in many sundials. On any given day, the sun revolves in a circle on the celestial sphere, and its rays striking the point on a sundial traces out a cone of light. The intersection of this cone with the horizontal plane of the ground forms a conic section. At most populated latitudes and at most times of the year, this conic section is a hyperbola.

Efficient portfolio frontier

In portfolio theory, the locus of mean-variance efficient portfolios (called the efficient frontier) is the upper half of the east-opening branch of a hyperbola drawn with the portfolio return's standard deviation plotted horizontally and its expected value plotted vertically; according to this theory, all rational investors would choose a portfolio characterized by some point on this locus.

Extensions

The three-dimensional analog of a hyperbola is a hyperboloid. Hyperboloids come in two varieties, those of one sheet and those of two sheets. A simple way of producing a hyperboloid is to rotate a hyperbola about the axis of its foci or about its symmetry axis perpendicular to the first axis; these rotations produce hyperboloids of two and one sheet, respectively.

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