International Journal of Research in Engineering and Applied Sciences (IJREAS)
Available online at http://euroasiapub.org/journals.php
Vol. 6 Issue 10, October - 2016, pp. 75~86
ISSN(O): 2249-3905, ISSN(P) : 2349-6525 | Impact Factor: 6.573 | Thomson Reuters ID: L-5236-2015

Inversion theorem for Laplace-Weierstrass transform

V. N. Mahalle[1],
Assistant Professor, Department of Mathematics,
Bar. R.D.I.K.N.K.D. College, Badnera Railway, Maharashtra, India

S.S. Mathurkar[2],
Assistant Professor, Department of Mathematics, Government College of Engineering, Amravati, Maharashtra, India

R. D. Taywade[3]
Assistant Professor, Department of Mathematics, Prof. Ram Meghe Institute of Technology & Research, Badnera, Amravati, Maharashtra, India

ABSTRACT: We know that for the application of an integral transform, the main condition is the validity of the inversion theorem which allows one to find an unknown function by knowing its image. Keeping this in view, in the present paper we have proved the inversion theorem by proving some lemmas required for the proof of inversion theorem for Laplace-Weierstrass transform. Also uniqueness theorem is proved.

Keywords: Laplace transform, Weierstrass transform, Laplace-Weierstrass transform, testing function space.

I. Introduction:

In mathematics the Laplace and Weierstrass transform are most important and useful integral transforms. These transforms takes a function of positive real variables t and y to a function of a complex variable s, x respectively. Laplace-Weierstrass transform are usually restricted to functions of t and y with $t, y > 0$. A consequence of this restriction is that the Laplace-Weierstrass transform of a function is a holomorphic function of the variables s and x. As a holomorphic function, the Laplace-Weierstrass transform has a power series representation.

The Laplace transform is invertible on a large class of functions. The inverse

Laplace transform takes a function of a complex variable s (frequency) and yields a function of a real variable t (time). Laplace transformation from the time domain to the frequency domain transforms differential equations into algebraic equations. It has many applications in the sciences and technology.

Bilodeau [1] proved inversion formula for the Weierstrass transform. Kene and Gudadhe [2] had presented the some properties of generalized Mellin Whittaker transform. Mathurkar *et.al* [3,4] discussed the analyticity of Laplace Weierstrass transform with elementary properties. Pathak [5] extended integral transform to the compact support. Robbin and Huang [6] developed inverse filtering for linear shift-variant imaging system. Thakur and Tamrakar [7] created convergence and inversion theorem for generalized Weierstrass transform. Zemanian [8] had studied integral transforms like Laplace, Mellin, Hankel, K, Weierstrass in distributional sense. Motivated by above, in this paper we have formed the inversion theorem for Laplace-Weierstrass transform.

This paper emphasizes as follows. In section II we defined testing function space for Laplace-Weierstrass transform. Section III gives three Lemmas required for inversion theorem In section IV inversion theorem & Uniqueness theorem for the same. And paper is concluded lastly in section V.

II. The Testing Function Space $LW_{a,b}$:

$LW_{a,b}$ as the linear space of all complex valued smooth functions $\phi(t,y)$ on $0<t<\infty$, $0<y<\infty$ such that for each $p, q = 0, 1, 2, ---$

$$\gamma_{a,b,p,q}\,\phi(t,y)=\sup_{\substack{0<t<\infty\\0<y<\infty}}\left|e^{at-\frac{by}{2}+\frac{y^2}{4}}D_t^p\,D_y^q\,\phi(t,y)\right|<\infty,$$

(2.1)

for some fixed numbers a,b in R

The space $LW_{a,b}$ is complete and a Frechet space. This topology is generated by the total families of countably multinorms space given by (2.1).
The proof of the inversion theorem requires following lemmas.

III. Lemmas:

3.1 Lemma 1:

Let $LW\{f(t,y)\}=F(s,x)$, for $\sigma_1<\operatorname{Re}s<\sigma_2$ and $\sigma_1'<\operatorname{Re}x<\sigma_2'$. Let $\phi(t,y)\in D(\Omega)$, where $(t,y)\in\Omega$ and set $\psi(s,x)=\frac{1}{\sqrt{4\pi}}\int_0^\infty\int_0^\infty\phi(t,y)e^{st+\frac{(x-y)^2}{4}}\,dy\,dt$. Then for any fixed real numbers r and r' with $0<r<\infty$ & $0<r'<\infty$

$$\frac{1}{\sqrt{4\pi}}\int_{-r}^{r}\int_{-r'}^{r'}\left\langle f(g,h),e^{-sg-\frac{(x-h)^2}{4}}\right\rangle\psi(s,x)\,d\eta\,d\xi=\frac{1}{\sqrt{4\pi}}\left\langle f(g,h),\int_{-r}^{r}\int_{-r'}^{r'}e^{-sg-\frac{(x-h)^2}{4}}\psi(s,x)\,d\eta\,d\xi\right\rangle$$

(3.1.1)

Where $s=\sigma+i\xi$ and $x=\sigma'+i\eta$, σ & σ' are any fixed real numbers such that $\sigma_1<\sigma<\sigma_2$ and $\sigma_1'<\sigma<\sigma_2'$.

Proof:- The case $\phi(t,y)=0$ is trivial. Let us consider $\phi(t,y)\neq 0$ for any $(t,y)\in\Omega$. Since $F(s,x)$ is analytic for $\sigma_1<Res<\sigma_2$ and $\sigma_1'<\operatorname{Re}x<\sigma_2'$ and $\psi(s,x)$ is an entire function, then the r. h. s. side integral of the equation (3.1.1) exists. We first show that, $v(g,h)=\frac{1}{\sqrt{4\pi}}\int_{-r}^{r}\int_{-r'}^{r'}e^{-sg-\frac{(x-h)^2}{4}}\psi(s,x)\,d\eta\,d\xi$, as a function of (g,h) which is belongs to $LW_{a,b}$. For this we consider,

$$\gamma_{a,b,p,q}\, \nu(g,h) = \sup_{\substack{0<g<\infty \\ 0<h<\infty}} \left| e^{ag-\frac{bh}{2}+\frac{h^2}{4}} D_g^p D_h^q\, \nu(g,h) \right|$$

$$= \sup_{\substack{0<g<\infty \\ 0<h<\infty}} \left| e^{ag-\frac{bh}{2}+\frac{h^2}{4}} D_g^p D_h^q \frac{1}{\sqrt{4\pi}} \int_{-r}^{r}\int_{-r'}^{r'} e^{-sg-\frac{(x-h)^2}{4}} \psi(s,x)\, d\eta\, d\xi \right|$$

Carrying the operator $D_g^p D_h^q$ within the integral and summation sign, which is easily justified

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due to smoothness of the integral, we get

$$\gamma_{a,b,p,q}\nu(g,h) = \sup_{\substack{0<g<\infty \\ 0<h<\infty}} \left| e^{ag-\frac{bh}{2}+\frac{h^2}{4}} \frac{1}{\sqrt{4\pi}} \int_{-r}^{r}\int_{-r'}^{r'} D_g^p D_h^q\, e^{-sg-\frac{(x-h)^2}{4}} \psi(s,x)\, d\eta\, d\xi \right|$$

$$=$$

$$\sup_{\substack{0<g<\infty \\ 0<h<\infty}} \left| \frac{1}{\sqrt{4\pi}} \int_{-r}^{r}\int_{-r'}^{r'} \psi(s,x)(-s)^p e^{-(s-a)g-\frac{[(x-h)^2-h^2]}{4}-\frac{bh}{2}} P_q(x-h)\, d\eta\, d\xi \right| \tag{3.1.2}$$

Where P_q is polynomial in *q*. The series in right hand side is series of positive finite terms which is bounded by *K* for *g*> 0 and *h*> 0. Therefore $\gamma_{a,b,p,q}\, \nu(g,h) < \infty$, hence $\nu(g,h)$ belongs to $LW_{a,b}$. Therefore r. h. s. of (3.1.1) is meaningful.

Next, partition the path of integration on the straight line from $s-ir$ to $s+ir$ into m intervals, each of length $\frac{2r}{m}$ and from $x=\sigma'-ir'$ to $x=\sigma'+ir'$ into n intervals each of length $\frac{2r'}{n}$.

Let $s_i = \sigma + ir_i$ be any point in the i^th^ interval and $x_j = \sigma' + ir_j'$ be any point in the j^th^ interval.

Consider,

$$\theta_{m,n}(g,h) = \frac{1}{\sqrt{4\pi}} \sum_{i=1}^{m}\sum_{j=1}^{n} \psi(s_i, x_j)\, e^{-s_i g-\frac{(x_j-h)^2}{4}} \left(\frac{4rr'}{mn}\right)$$

$$= \frac{1}{\sqrt{\pi}} \sum_{i=1}^{m}\sum_{j=1}^{n} \psi(s_i, x_j)\, e^{-s_i g-\frac{(x_j-h)^2}{4}} \left(\frac{2rr'}{mn}\right)$$

(3.1.3)

Operate $f(g,h)$ to above equation (3.1.3) term by term, we get

$$\left\langle f(g,h),\theta_{m,n}(g,h)\right\rangle=\left\langle f(g,h),\frac{1}{\sqrt{\pi}}\sum_{i=1}^{m}\sum_{j=1}^{n}\psi(s_i,x_j)\,e^{-s_i g-\frac{(x_j-h)^2}{4}}\,\frac{2r\,r'}{mn}\right\rangle$$

$$=\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{1}{\sqrt{\pi}}\left\langle f(g,h),e^{-s_i g-\frac{(x_j-h)^2}{4}}\right\rangle\psi(s_i,x_j)\frac{2r\,r'}{mn} \qquad (3.1.4)$$

In view of the fact that $\frac{1}{\sqrt{\pi}}\left\langle f(g,h),e^{-s_i g-\frac{(x_j-h)^2}{4}}\right\rangle\psi(s_i,x_j)$ is continuous function on $-r\le\xi\le r$ and $-r'\le\eta\le r'$, the sum on the right hand side of equation (3.1.4) tends to l. h. s. of equation (3.1.1) as $m,n\to\infty$.

So we need to show that, for each fixed p and q, $\left|\theta_{m,n}(g,h)-\nu(g,h)\right|$ converges uniformly to zero on $0<g<\infty$ and $0<h<\infty$ as $m\to\infty$ and $n\to\infty$.

Consider,

$$\gamma_{a,b,p,q}\left[\theta_{m,n}(g,h)-\nu(g,h)\right]=\sup_{\substack{0<g<\infty\\ o<h<\infty}}\left|e^{ag-\frac{by}{2}+\frac{h^2}{4}}\,D_g^p\,D_h^q\left[\theta_{m,n}(g,h)-\nu(g,h)\right]\right|$$

$$=\sup_{\substack{0<g<\infty\\ 0<h<\infty}}\left|\begin{array}{l}\frac{(-1)^p}{\sqrt{\pi}}e^{ag-\frac{bh}{2}+\frac{h^2}{4}}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{q=0}^{k}\psi(s_i,x_j)(s_i)^p\,P_q(x_j-h)e^{-s_i g-\frac{(x_j-h)^2}{4}}\,\frac{2r\,r'}{mn}-\frac{(-1)^p}{\sqrt{4\pi}}e^{ag-\frac{bh}{2}+\frac{h^2}{4}}\\ \int_{-r}^{r}\int_{-r'}^{r'}\sum_{q=0}^{k}(s)^p\,P_q(x-h)e^{-sg-\frac{(x-h)^2}{4}}\,\psi(s,x)\,d\eta\,d\xi\end{array}\right| \qquad (3.1.5)$$

Where P_q is the polynomial in q and q is finite quantity.

Notice that

$$\left|\sum_{q=0}^{k}e^{ag-\frac{bh}{2}+\frac{h^2}{4}-sg-\frac{(x-h)^2}{4}}\right|\to 0 \text{ as } g\to\infty \text{ and } h\to\infty \text{ for } \operatorname{Re}s>a \text{ and } \operatorname{Re}x>b.$$

Therefore, given $\in > 0$, we can choose G and H so large that for all $g > G \text{ and } h > H$,

$$\left| \sum_{q=0}^{k} \frac{1}{\sqrt{4\pi}} e^{ag - \frac{bh}{2} + \frac{h^2}{4} - sg - \frac{(x-h)^2}{4}} \right| < \frac{\in}{3} \left[\int_{-r}^{r} \int_{-r'}^{r'} \left| \sum_{q=0}^{k} (s)^p P_q(x-h) \psi(s,x) \right| d\eta \, d\xi \right]^{-1}$$

Which is finite quantity. Since $\varphi(t,y) \neq 0$ the right hand side is finite.

Now for all $g > G \text{ and } h > H$, the magnitude of the second term on the right hand side of equation (3.1.5) is bounded by $\frac{\in}{3}$.

Moreover again for *g* >*G* and *h* >*H*, the magnitude of the first term on the right hand side of equation (3.1.5) is bounded by

$$\frac{\in}{3} \left[\int_{-r}^{r} \int_{-r'}^{r'} \left| \sum_{q=0}^{k} (s)^p P_q(x-h) \psi(s,x) \right| d\eta \, d\xi \right]^{-1} \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{q=0}^{k} \left| (s_i)^p P_q(x_j - h) \psi(s_i, x_j) \right| \frac{2r r'}{mn}$$

(3.1.6)

Now choose $m_0' \,\&\, n_0'$, so large that, for $m > m_0' \text{ and } n > n_0'$, the above expression (3.1.6) is less than $\frac{2\in}{3}$. Thus, for all $g > G \text{ and } h > H$ and $m > m_0' \text{ and } n > n_0'$, the r. h. s. of equation (3.1.5) is less than $\in$ i.e. $\gamma_{a,b,p,q}\left[\theta_{m,n}(g,h) - \nu(g,h)\right] < \in$. Moreover, on $0 \le g \le G,\ 0 \le h \le H$ and $-r \le \xi \le r,\ -r' \le \eta \le r'$, the expression

$$\frac{1}{\sqrt{4\pi}} e^{ag - \frac{bh}{2} + \frac{h^2}{4}} \sum_{q=0}^{k} (s)^p P_q(x-h) e^{-sg - \frac{(x-h)^2}{4}} \psi(s,x)$$

is uniformly continuous function. Therefore, in view of equation (3.1.6), there exists an $m_1' \text{ and } n_1'$ such that for all $m > m_1' \text{ and } n > n_1'$, $\gamma_{a,b,p,q}\left[\theta_{m,n}(g,h) - \nu(g,h)\right] < \in$ on $0 \le g \le G \text{ and } 0 \le h \le H$ as well. Thus when $m > \max(m_0', m_1')$ and $n > \max(n_0', n_1')$,

$\gamma_{a,b,p,q}\left[\theta_{m,n}(g,h) - \nu(g,h)\right] < \in$ on $0 < g < \infty$ and $0 < h < \infty$.

This completes the proof.

3.2 Lemma 2:

For $\phi(t,y) \in D$, set $\psi(s,x) = \frac{1}{\sqrt{4\pi}} \int_0^\infty \int_0^\infty \phi(t,y) e^{st + \frac{(x-y)^2}{4}} dy\, dt$ as in lemma (1) then,

$$\frac{1}{\sqrt{4\pi}} \int_{-r}^{r} \int_{-r'}^{r'} e^{-sg - \frac{(x-h)^2}{4}} \psi(s,x)\, d\eta\, d\xi = \frac{1}{\pi} \int_0^\infty \int_0^\infty e^{\frac{y^2}{4} - \frac{h^2}{4}} \phi(t,y) e^{\sigma(t-g)} \frac{\sin r(t-g)}{(t-g)} e^{\sigma'} \frac{(h-y)}{2} \frac{\sin r'\left(\frac{h-y}{2}\right)}{\left(\frac{h-y}{2}\right)} dy\, dt$$

Proof: We shall prove the result by justifying the steps in the following manipulations and by considering compact support of $\varphi(t,y) \in D_a$

$$\frac{1}{\sqrt{4\pi}} \int_{-r}^{r} \int_{-r'}^{r'} e^{-sg - \frac{(x-h)^2}{4}} \psi(s,x)\, d\eta\, d\xi = \frac{1}{\sqrt{4\pi}} \int_{-r}^{r} \int_{-r'}^{r'} e^{-sg - \frac{(x-h)^2}{4}} \left[\frac{1}{\sqrt{4\pi}} \int_0^\infty \int_0^\infty \phi(t,y) e^{st + \frac{(x-y)^2}{4}} dy\, dt \right] d\eta\, d\xi$$

$$\frac{1}{\sqrt{4\pi}} \int_{-r}^{r} \int_{-r'}^{r'} e^{-sg - \frac{(x-h)^2}{4}} \psi(s,x)\, d\eta\, d\xi = \frac{1}{\pi} \int_0^\infty \int_0^\infty e^{\frac{y^2}{4} - \frac{h^2}{4}} \phi(t,y) e^{\sigma(t-g)} \frac{\sin r(t-g)}{(t-g)} e^{\sigma'} \frac{(h-y)}{2} \frac{\sin r'\left(\frac{h-y}{2}\right)}{\left(\frac{h-y}{2}\right)} dy\, dt$$

3.3 Lemma 3:

Let $a_1, a_2, b_1, b_2, \sigma, \sigma', r$ and r' be real numbers with $a_1 < \sigma < a_2$ and $b_1 < \sigma' < b_2$. Also let $\phi(t,y) \in D$.

Then, $$\frac{1}{\pi} \int_0^\infty \int_0^\infty e^{\frac{y^2}{4} - \frac{h^2}{4}} \phi(t,y) e^{\sigma(t-g)} \frac{\sin r(t-g)}{(t-g)} e^{\sigma'} \frac{(h-y)}{2} \frac{\sin r'\left(\frac{h-y}{2}\right)}{\left(\frac{h-y}{2}\right)} dy\, dt = C(g,h)$$

Converges in $LW_{a,b}$ to $\phi(g,h)$ as $r, r' \to \infty$.

Proof: Here we have to prove that

$$\frac{1}{\pi}\int_0^\infty\int_0^\infty e^{\frac{y^2}{4}-\frac{h^2}{4}}\phi(t,y)e^{\sigma(t-g)}\frac{\sin r(t-g)}{(t-g)}e^{\sigma'}\frac{(h-y)}{2}\frac{\sin r'\left(\frac{h-y}{2}\right)}{\left(\frac{h-y}{2}\right)}dy\,dt \to \phi(g,h)$$

For this we show that

$$\gamma_{a,b,p,q}\left|C(g,h)-\phi(g,h)\right| \to 0$$

Now consider,

$$\gamma_{a,b,p,q}\left|C(g,h)-\phi(g,h)\right|$$

$$= \sup_{\substack{0<g<\infty\\0<h<\infty}}\left|e^{ag-\frac{bh}{2}+\frac{h^2}{4}}D_g^p\,D_h^q\left[\frac{1}{\pi}\int_0^\infty\int_0^\infty e^{\frac{y^2}{4}-\frac{h^2}{4}}\phi(t,y)e^{\sigma(t-g)}\frac{\sin r(t-g)}{(t-g)}e^{\sigma'}\frac{(h-y)}{2}\frac{\sin r'\left(\frac{h-y}{2}\right)}{\left(\frac{h-y}{2}\right)}dy\,dt\right]-\phi(g,h)\right| \to 0$$

$$\frac{1}{\pi}\int_0^\infty\int_0^\infty \phi(t,y)e^{\sigma(t-g)}\frac{\sin r(t-g)}{(t-g)}e^{\sigma'}\frac{(h-y)}{2}\frac{\sin r'\left(\frac{h-y}{2}\right)}{\left(\frac{h-y}{2}\right)}dy\,dt = \phi(g,h)$$

(3.3.1)

(3.3.2)

Using equation (3.3.2) in equation (3.3.1), we get

$$\gamma_{a,b,p,q}\left|C(g,h)-\phi(g,h)\right|$$

$$= \sup_{\substack{0<g<\infty\\0<h<\infty}}\left|e^{ag-\frac{bh}{2}+\frac{h^2}{4}}D_g^p\,D_h^q\left[\begin{array}{l}\frac{1}{\pi}\int_0^\infty\int_0^\infty e^{\frac{y^2}{4}-\frac{h^2}{4}}\phi(t,y)e^{\sigma(t-g)}\frac{\sin r(t-g)}{(t-g)}e^{\sigma'}\frac{(h-y)}{2}\frac{\sin r'\left(\frac{h-y}{2}\right)}{\left(\frac{h-y}{2}\right)}dy\,dt\\ -\frac{1}{\pi}\int_0^\infty\int_0^\infty \phi(t,y)e^{\sigma(t-g)}\frac{\sin r(t-g)}{(t-g)}e^{\sigma'\left(\frac{h-y}{2}\right)}\frac{\sin r'\left(\frac{h-y}{2}\right)}{\left(\frac{h-y}{2}\right)}dy\,dt\end{array}\right]\right|$$

$$= \sup_{\substack{0<g<\infty \\ 0<h<\infty}} \left| e^{ag-\frac{bh}{2}+\frac{h^2}{4}} D_g^p D_h^q \left[\frac{-2}{\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{y^2}{4}-\frac{h^2}{4}} \phi(t_1+g, h-2y_1) e^{\sigma t_1} \frac{\sin rt_1}{t_1} e^{\sigma' y_1} \frac{\sin r' y_1}{y_1} dy_1\, dt_1 + \frac{2}{\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi(t_1+g, h-2y_1) e^{\sigma t_1} \frac{\sin rt_1}{t_1} e^{\sigma' y_1} \frac{\sin r' y_1}{y_1} dy_1\, dt_1 \right] \right|$$

As A. H. Zemanian [1] pp. 66 and theorem 3.5.1, we can write

$$\gamma_{a,b,p,q} \left| C(g,h) - \phi(g,h) \right| \to 0.$$

This completes the proof.

IV Theorem:

4.1 Inversion Theorem:

Let $F(s,x) = LW\{f(t,y)\}$ for $s, x \in \Omega_f$. Also let r and r' are real variables. Then in the sense of convergence in D'

$$f(t,y) = \lim_{r,r' \to \infty} \frac{-1}{4\pi\sqrt{\pi}} \int_{\sigma-ir}^{\sigma+ir} \int_{\sigma'-ir'}^{\sigma'+ir'} F(s,x) e^{st+\frac{(x-y)^2}{4}} dx\, ds \qquad (4.4.1)$$

Where σ and σ' are any fixed positive real nos. belonging to Ω_f, $\sigma_1 < \sigma < \sigma_2$ and $\sigma_1' < \sigma < \sigma_2'$.

Proof: We need to show that for $\phi(t,y) \in D(\Omega)$.

$$\left\langle \frac{-1}{4\pi\sqrt{\pi}} \int_{\sigma-ir}^{\sigma+ir} \int_{\sigma'-ir'}^{\sigma'+ir'} F(s,x) e^{st+\frac{(x-y)^2}{4}} dx\, ds,\ \varphi(t,y) \right\rangle = \frac{-1}{2\pi} \langle f(g,h), \phi(g,h) \rangle \qquad (4.4.2)$$

$$as\ r, r' \to \infty$$

From the analyticity of $F(s,x)$ on Ω_f and the fact that the $\phi(t,y)$ has compct support

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in D(Ω) it follows that the left hand side equation in (4.4.2) is merely a repeated integral with respect to *t, s, y* and *x* and that the above integrand is continuous on the closed bounded domain of integration. Therefore, the left hand side without the limit notation can be written as,

$$\int_0^\infty\int_0^\infty \phi(t,y)\left[\frac{-1}{4\pi\sqrt{\pi}}\int_{\sigma-ir}^{\sigma+ir}\int_{\sigma'-ir'}^{\sigma'+ir'} F(s,x)e^{st+\frac{(x-y)^2}{4}}\,dx\,ds\right]dy\,dt$$

Letting $s=\sigma+i\xi$ and $x=\sigma'+i\eta$ we get,

$$\frac{-1}{4\pi\sqrt{\pi}}\int_0^\infty\int_0^\infty \phi(t,y)\int_{-r}^{r}\int_{r'}^{r'} F(s,x)e^{st+\frac{(x-y)^2}{4}}\,d\eta\,d\xi\,dy\,dt$$

Since Φ(t,y) has a compact support and integrand is a continuous function of (t,y,ξ,η), the

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order of integration may be changed. These yields

$$\frac{-1}{4\pi\sqrt{\pi}}\int_{-r}^{r}\int_{-r'}^{r'} F(s,x)\int_0^\infty\int_0^\infty \phi(t,y)e^{st+\frac{(x-y)^2}{4}}\,dy\,dt\,d\eta\,d\xi$$

$$=\frac{-1}{4\pi\sqrt{\pi}}\int_{-r}^{r}\int_{-r'}^{r'}\frac{1}{\sqrt{4\pi}}\left\langle f(g,h), e^{-sg-\frac{(x-h)^2}{4}}\right\rangle\int_0^\infty\int_0^\infty \phi(t,y)e^{st+\frac{(x-y)^2}{4}}\,dy\,dt\,d\eta\,d\xi$$

$$=\left\langle f(g,h), \frac{-1}{2\pi}\left[\frac{1}{\pi}\int_0^\infty\int_0^\infty e^{\frac{y^2}{4}-\frac{h^2}{4}}\phi(t,y)e^{\sigma(t-g)}\frac{\sin r(t-g)}{(t-g)}e^{\sigma'}\frac{(h-y)}{2}\frac{\sin r'\left(\frac{h-y}{2}\right)}{\left(\frac{h-y}{2}\right)}dy\,dt\right]\right\rangle$$

Because f belongs to $LW_{a,b}$ and in view of lemma 2 & 3, the last expansion tends to

$$\left\langle f(g,h), \frac{-1}{2\pi}\phi(g,h)\right\rangle$$

$$=\frac{-1}{2\pi}\langle f(g,h), \phi(g,h)\rangle$$

Which complete the proof of the inversion theorem.

4.2 Uniqueness Theorem:

Let $LW\{f(t,y)\}=F(s,x)$ for $(s,x)\in\Omega_f$ and $LW\{u(t,y)\}=U(s,x)$ for $(s,x)\in\Omega_u$ and if

$\Omega_f \cap \Omega_u$ is not empty. If $F(s,x)=U(s,x)$ on $\Omega_f \cap \Omega_u$, then $f(t,y)=u(t,y)$ in the sense of equality in $D'(\Omega)$.

Proof: For any $(t,y)\in\Omega$ and using inversion theorem, we get

$$\left\langle f(t,y)-u(t,y),\, \phi(t,y)\right\rangle = \left\langle \lim_{r,r'\to\infty} \frac{-1}{4\pi\sqrt{\pi}} \int_{\sigma-ir}^{\sigma+ir} \int_{\sigma'-ir'}^{\sigma'+ir'} \{F(s,x)-U(s,x)\} e^{st+\frac{(x-y)^2}{4}}\, dx\, ds , \phi(t,y) \right\rangle$$

(4.2.1)

But given that $F(s,x)=U(s,x)$

Therefore equation (4.2.1) becomes

$$=0, \quad (t,y)\in D(\Omega)$$

Hence $f(t,y)=u(t,y)$ in the sense of equality in $D'(\Omega)$.

V. Conclusion:

This paper concludes inversion theorem by using some lemmas and uniqueness theorem for Laplace-Weierstrass transform.

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