# Generalized Euler Formula For Curvature

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## Abstract

This study aims to show how to obtain the curvature of the ellipsoid for generalized surfaces. The curvature topic is quite popular at an interdisciplinary level. It can be to the friends of geometry,geodesy,satellite orbits in space, in studying all sorts of elliptical motions (e.g., planetary motions), curvature of surfaces and concerning eye-related radio-therapy treatment, for example the anterior surface of the cornea is often represented as ellipsoidal in form.On the calculation of the curvature, there is a famous Euler formula for rotating ellipsoid that everyone knows.I wonder how can a formula for the general surfaces (e.g. hyperboloid, triaxial ellipsoid)? so we started to work. In a way I generalize the Euler formula, and I gave the original a formula for it.

**Keywords:**Curvature, Triaxial Ellipsoid, Normal Section Curve, Principal Curvatures, Bektas Formula

## 1.Introduction

The curvature issue is very important in geodesy and also inophthalmology. To make geodetic computations on the ellipsoid (rotational or triaxial) first we need to know the normal section curve that combines observation points. The normal section curve is also available from the intersection of the ellipsoid and a plane which contains normal of surface on the station point and passes destination point. Geodetic computational formulas are contain the curvature parameters.This current study aims to pave the way for our further study on triaxial ellipsoid work.

Considerable several numbers of relevant studies were found in the literature. Some of them, Harris 2006,Lipschutz 1969, Barbero 2015,Bennett 1988,Douthwaite et al 1998. Concerning the study ofHarris in 2006, I think he was made a mistake. Curvature calculation was produced based on the Cartesian coordinates. However, the curvature calculation should have been based on the surface parameters (*u,v* parameters) not on the Cartesian coordinates.

When we look at the literature, we see that the curvature calculation is usually given depending on the angle of parameters. However, in practice, the azimuth angle is used instead of the angle of parametersand azimuth angle can be easily calculated from the Cartesian coordinates. At this point the importance of our study appears. We give the curvature calculation depending on the angle of the ellipsoid azimuth with a new formula. And I think it has not been previously in the related.On the other hand, we also notice the lack of numerical practical studies in the literature; and therefore, we have added a comprehensive numeric application to our work.

## 2.General surfaces

A point on surface can be represented by means of its position vector relative to the origin of an orthogonal coordinate system:

*X=(x,y,z)*

Suppose each of the coordinates is a function of two parameters *u,v*

$x = x(u, v)$
$y = y(u, v)$
$z = z(u, v)$

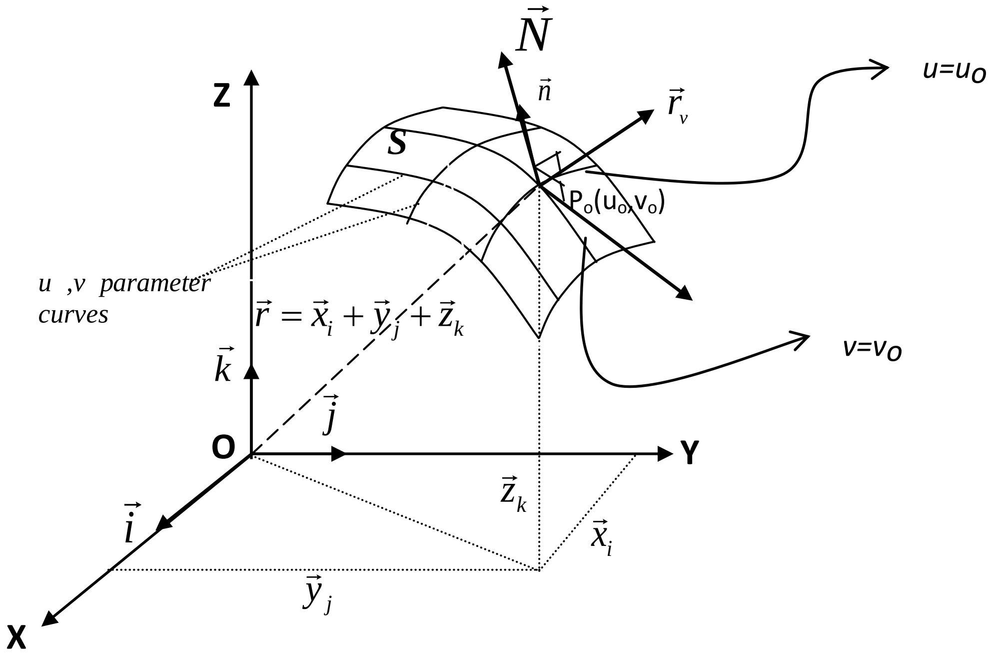


*Figure-1 Parametric representation of surfaces S*

### 2.1 Ellipsoid

An ellipsoid is a closed quadric surface that is analogue of an ellipse. Ellipsoid has three different axes (a>b>c) as shown in Fig.2.Mathematical literature often uses “ellipsoid” in place of “Triaxial ellipsoid or general ellipsoid”. Scientific literature (particularly geodesy) often uses “ellipsoid” in place of “biaxial ellipsoid, rotational ellipsoid or ellipsoid revolution” (a = b > c). Older literature uses ‘spheroid’ in place of rotational ellipsoid. The standard equation of an ellipsoid centered at the origin of a Cartesian coordinate system and aligned with the axes .General ellipsoid equation as below Bektas, 2014.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0 \quad (1)$$

The following definitions will be used.

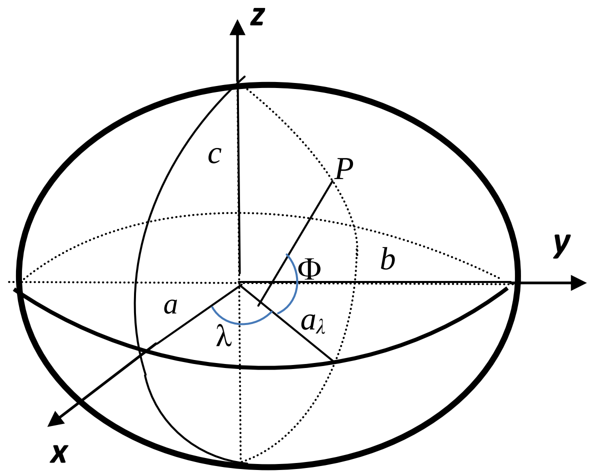


*Figure-2 Triaxial Ellipsoid*

$a$ = equatorial semi majoraxis of the ellipsoid
$b$ = equatorial semi minor axis of the ellipsoid
$c$ = polar semi-minor axis of the ellipsoid
$\lambda$ = geodetic longitude
$\Phi$ = geodetic latitude
$h$ = ellipsoid height
$e_x^2 = (a^2-c^2)/a^2$ first polar eccentricity
$e_e^2 = (a^2-b^2)/a^2$ first equatorial eccentricity

The differences between Rotational Elipsoid and Triaxial EllipsoidBektas,2015"b":

-In Rotational ellipsoid ,all latitude circles ($\Phi$= constant) are circle geometry whereas in Triaxial ellipsoid, all latitude circles are ellips geometry. But one in any case,these lines are not plane lines.

-In rotational ellipsoid ,all longitude circles ($\lambda$=constant) are the same ellips geometry, Ellips's semi axis ($a,b$) equals to rotational ellipsoid. Whereas in Triaxial ellipsoid, all longitude circles are different ellips geometry. These ellips semi axis are ($a_\lambda$, $c$) but semi-major axis $a_\lambda$ is changeabledepending on longitude($a \le a_\lambda \le b$)

$$a_\lambda = a\sqrt{\frac{\cos^2\lambda + (1-e_e^2)^2\sin^2\lambda}{1-e_e^2\sin^2\lambda}} \quad (2)$$

$\lambda$ meridian ellipse's semi major-axis

Ellipsoid equation (u,v) Gauss Parametric form

$$x = a\cos u \sin v$$
$$y = b \sin u \sin v \quad (3)$$
$$z = c \cos v$$

$-\pi/2 \le u \le \pi/2, \quad -\pi \le v \le \pi$

Parametric coordinates calculated from Cartesian coordinates as below formula

$$v = \arccos\left(\frac{z}{c}\right)$$
$$u = \arctan\left(\frac{a.y}{b.x}\right) \quad (4)$$

The parameterlines *(u,v*) and geodetic (planetographic) coordinates (Φ, λ) are orthogonal on rotational ellipsoid but are not orthogonal on triaxial ellipsoid.

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In this parametrization, the coefficients of the first fundamental form are

$E=[b^2cos^2 u + a^2sin^2 u] sin^2 v$ (5)
$F=(b^2-a^2) cos u sin u cos v sin v$ (6)
$G=[a^2 cos^2u + b^2sin^2 u] cos^2 v + c^2 sin^2 v$ (7)

I. fundamental form $I = E.du^2 + 2.F.du.dv + G.dv^2$ (8)

and of the second fundamental form are

$$e = \frac{a.b.c.\sin^2 v}{\sqrt{(a.b.\cos v)^2 + c^2(b^2\cos^2 u + a^2\sin^2 u)\sin^2 v}} \quad (9)$$

$f=0$ (10)

$$g = \frac{a.b.c}{\sqrt{(a.b.\cos v)^2 + c^2(b^2\cos^2 u + a^2\sin^2 u)\sin^2 v}} \quad (11)$$

II. fundamental form $II = e.du^2 + 2.f.du.dv + g.dv^2$ (12)
Also in this parametrization, the Gaussian curvature is

$$K = \left[\frac{a.b.c}{(a.b.\cos v)^2 + c^2(b^2\cos^2 u + a^2\sin^2 u)\sin^2 v}\right]^2 \quad (13)$$

and the Mean curvature is

$$H = \frac{a.b.c.[3(a^2+b^2) + 2c^2 + (a^2+b^2-2c^2)\cos 2v - 2(a^2-b^2)\cos 2u.\sin^2 v]}{8[(a^2b^2\cos^2 v + c^2(b^2\cos^2 u + a^2\sin^2 u)\sin^2 v]^{3/2}} \quad (14)$$

The Gaussian curvature and Mean curvaturecan be calculated from Cartesian coordinates given below formulasLipschutz 1969, Zhangand Feng 2006

$$.K = \frac{1}{\left(\mathrm{a.b.c.}\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)\right)^2} \quad (15)$$

$$H = \frac{\left|x^2 + y^2 + z^2 - a^2 - b^2 - c^2\right|}{2(a.b.c)^2\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)^{3/2}} \qquad (16)$$

We will compute H and K in terms of the first and the second fundamental form.

$$K = \frac{e.g - f^2}{E.G - F^2} = \frac{\mathrm{II}}{\mathrm{I}} \qquad (17)$$

$$H = \frac{G.e - 2F.f + E.g}{2(E.G - F^2)} \qquad (18)$$

## 2.2 Principal Curvatures, Gaussian Curvature, Mean Curvature

We will now study how the normal curvature at a point varies when a unit tangent vector varies. In general, we will see that the normal curvature has a minimum value $\kappa_1$ and a maximum value $\kappa_2$,. This was shown by Euler in 1760. The quantity

$K = \kappa_1.\kappa_2$ called the Gaussian curvature (19)

and the quantity

$H = (\kappa_1 + \kappa_2)/2$called the mean curvature, (20)

play a very important role in the theory of surfaces.

$$R_1 = \frac{1}{\kappa_1} = \frac{1}{H - \sqrt{H^2 - K}} \qquad \text{Maximum radii of curvature} \qquad (21)$$

$$R_2 = \frac{1}{\kappa_2} = \frac{1}{H + \sqrt{H^2 - K}} \qquad \text{Minimum radii of curvature} \qquad (22)$$

The formula for the radius of curvature at arbitrary azimuth points up that the fact that the fundamental mathematical quantity is the inverse of these radii, which are simply called curvatures

## 2.3 The angle between the parameter curves

The angle θ is between *u* and *v* parameters curves that intersect at $P_o$ point. θis equal the angle between those tangent vectors $r_u$and$r_v$ at $P_o$ point the same time are surface tangent. This θ angle is found scalar product of the tangent vector $r_u$and$r_v$ as follows.

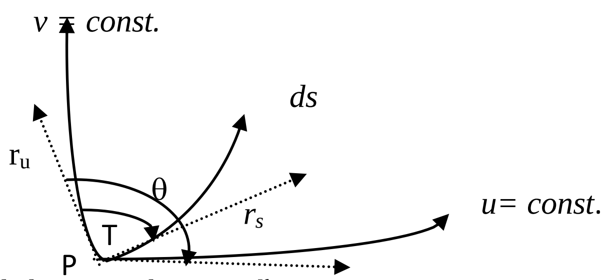


*Figure-3The angle between the parameter curves*

$\overrightarrow{r_u}\,\overrightarrow{r_v} = |\,r_u\,|\,|\,r_v\,|\cos\theta$ (23)

$$\cos\theta = \frac{r_u r_v}{|r_u|\,|\,r_v|} = \frac{F}{\sqrt{EG}} \qquad (24)$$

if $u \perp v$ $\theta = 90°$ and $\cos\theta = 0$ thus F became zero

$$\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{\frac{EG-F^2}{EG}} = \frac{W^*}{\sqrt{EG}} \quad (25)$$

where $W^* = \sqrt{EG-F^2}$

**Finding the T Angle** : T angle between any parameter curves in the figure above, for example *u ( v = const )* parameter curve and a any surface curve (s) is finding by the scalar product of the tangent vectors ($r_u$ *and* $r_s$). The tangent vector of the surface parameters curve (s) is $r_s = \frac{dr}{ds}$

$$\overrightarrow{r_u}\, r_s = |\overrightarrow{r_u}|\,|r_s| \cos T \quad (26)$$

$$\cos T = \frac{r_u r_s}{|r_u|\,|r_s|} \qquad |r_u| = \sqrt{E} \qquad |r_s| = 1 \quad (27)$$

$$|dr| = ds = \sqrt{r_u^2 du^2 + 2 r_u r_v\, du..dv + r_v^2 dv^2} \quad (28)$$

$$\frac{dr}{ds} = r_s = \frac{\partial r}{\partial u}\frac{du}{ds} + \frac{\partial r}{\partial v}\frac{dv}{ds} = r_u \frac{du}{ds} + r_v \frac{dv}{ds} \quad (29)$$

$$|r_u| = \sqrt{r_u^2} = \sqrt{E} \quad (30)$$

$$r_u r_v = F \quad (31)$$

$$|r_v| = \sqrt{r_v^2} = \sqrt{G} \quad (32)$$

$$ds_u = \sqrt{E}\, du \quad (33)$$

$$ds_v = \sqrt{G}\, dv \quad (34)$$

Here we create a product $r_u$ *and* $r_s$

$$r_u r_s = r_u r_u \frac{du}{ds} + r_u r_v \frac{dv}{ds} \quad (35)$$

$$r_u r_s = E\frac{du}{ds} + F\frac{dv}{ds} \quad (36)$$

If we put these values in terms of Tangle equality

$$\cos T = \frac{Edu + Fdv}{\sqrt{E}ds} \qquad \sin T = \frac{\sqrt{EG-F^2}dv}{\sqrt{E}ds} \quad \tan T = \frac{\sqrt{EG-F^2}dv}{Edu+Fdv} \quad (37)$$

If the parameter curves are orthogonal F will be zero (F = 0)

$$\cos T = \frac{\sqrt{E}du}{ds} = \frac{ds_{(u)}}{ds} \qquad \sin T = \frac{\sqrt{G}dv}{ds} = \frac{ds_{(v)}}{ds} \quad \tan T = \sqrt{\frac{G}{E}}\,\frac{dv}{du} = \frac{ds_{(v)}}{ds_{(u)}} \tag{38}$$

## 2.4 Normal section of a surface

Let us construct a normal to a surface at a point $P_o$. Then the curve that is described on the surface by any plane passing through the normal (i.e. containing the normal) is called a **normal section** of the surface(Fig.1). In other words a normal section is a plane section formed by a plane containing a normal to the surface Lipschutz(1969),James 1992,Aleksandrov et al 1999.

## 2.5 Curvature of a surface at a point

Let us construct a unit normal $\overline{n}$ and a tangent plane at given point $P_o$ on surface and consider the curves that are formed on the surface by planes passing through $P_o$ containing the normal i.e. the various normal sections passing through point $P_o$. Each normal section passing through $P_o$ possesses a particular curvature at point $P_o$. We can specify a particular normal section by use of a polar coordinate system constructed on the tangent plane, origin at point $P_o$, polar axis as some arbitrarily chosen initial ray in the tangent plane, and an angle α measured clockwise from the polar axis to the plane of the normal section (Fig.4). The curvature at point $P_o$ in direction α is thus given as the function $\kappa_n(\alpha)$. For each value of α there is a curvature associated with that particular normal section. This curvature $\kappa_n(\alpha)$ is called the **normal curvature** of the surface at point $P_o$ in the direction α.

Then the normal curvature at point $P_o$ is given by

$$\kappa_n(du,dv) = \frac{e.du^2 + 2.f.du.dv + g.dv^2}{E.du^2 + 2.F.du.dv + G.dv^2} \tag{39}$$

where E, F, G, e, f, g are the fundamental coefficients of the first and second order.

Formula (39) above can be re-written in the following way

$$\kappa_n\left(\frac{dv}{du}\right) = \frac{e + 2.f.\left(\frac{dv}{du}\right) + g.\left(\frac{dv}{du}\right)^2}{E + 2.F.\left(\frac{dv}{du}\right) + G.\left(\frac{dv}{du}\right)^2} \tag{40}$$

simply by dividing the numerator and denominator by *du²*. In this form it is obvious that $\kappa_n$ is a function of the ratio *dv/du*. If we let *cot* α *= dv/du*then (40) becomes

$$\kappa_n(\alpha) = \frac{e + 2.f.\cot\alpha + g.\cot^2\alpha}{E + 2.F.\cot\alpha + G.\cot^2\alpha} \tag{41}$$

where

$$\cot\alpha = \frac{E + F\tan\theta}{W\tan\theta} \tag{42}$$

A surface may be curved in many ways and consequently one might think that the dependence of the curvature $\kappa$ on the angle α might be arbitrary. In fact this is not so. The following theorem is due to Euler.

### 2.6 Euler's theorem

Let θ be the angle, in the tangent plane, measured clockwise from the direction of minimum curvature $\kappa_1$ . Then the normal curvature $\kappa_n(\theta)$ in direction θ is given by

$$\kappa_n(\theta) = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta = \kappa_1 + (\kappa_2 - \kappa_1) \cos^2\theta \qquad (43)$$

$\kappa_n(\theta)$ curvature at azimuth θ
For spheroid(rotational ellipsoid):

N = $R_1 = \dfrac{1}{\kappa_1}$ Radius of Curvature in Prime Vertical (Max. radii of curvature) (44)

M = $R_2 = \dfrac{1}{\kappa_2}$ Radius of Curvature in Meridian (Min. radii of curvature) (45)

N, M can be easily calculated the latitude of point $P_o$ as below

$$N= c_p / V , \quad M = c_p / V^3 , \qquad c_p = a^2/ c , \quad V^2=1+ e_x^2 \cos^2\Phi_o \qquad (46)$$

$$\kappa_n(\theta) = \frac{N.\cos^2\theta + M.\sin^2\theta}{N.M} \qquad \text{sameas Eq.43} \qquad (47)$$

$$\cot\alpha = \sqrt{\frac{E}{G}}\cot\theta \qquad (48)$$

### 2.7 Computing the principal directions and curvatures at a point $P_o$

Given a point $P_o$ on a surface ***S***, the directions at which the normal curvature at $P_o$ attains its minimum and maximum values can be computed as follows. Let the normal curvature at $P_o$ be given as

$$\kappa_n(\lambda) = \frac{e + 2.f.\lambda + g.\lambda^2}{E + 2.F.\lambda + G.\lambda^2} \qquad (49)$$

where λ = dv/du . We wish to find those values of λ at which the function $\kappa_n(\lambda)$ has its minimum and maximum values. We are thus faced with a problem of finding the maxima and minima of a function. A necessary condition for the function $\kappa_n(\lambda)$ to have a maxima or minima at a point is that at that point d $\kappa_n(\lambda)$ /dλ = 0. Using the usual formula for computing the derivative of a quotient we obtain

The directions corresponding to the minimum and maximum values of curvature are called the **principal directions** of the surface. The values $\kappa_1$ and $\kappa_2$ are called the **principal curvatures** of the surface James 1992,Aleksandrov et al 1999.

$$\frac{d\kappa_n(\lambda)}{d\lambda} = \frac{(E + 2.F.\lambda + G.\lambda^2)(2f + 2.g.\lambda) - (e + 2.f.\lambda + g.\lambda^2)(2.F + 2G.\lambda)}{(E + 2.F.\lambda + G.\lambda^2)^2} \qquad (50)$$

$$(E + 2F\lambda + G\lambda^2)(f + g\lambda) - (e + 2f\lambda + g\lambda^2)(F + G\lambda) = 0 \qquad (51)$$

For spheroid upon expansion f=0 and rearrangement(51) becomes

$$(Fg)\lambda^2 + (E g - G e)\lambda - F e = 0 \qquad (52)$$

One can then solve (51) or (52) for its two roots using the quadratic formula thus obtaining the two principal directions $\lambda_1$ and $\lambda_2$. One can then substitute the two values $\lambda_1$ and $\lambda_2$ into (49) to obtain the principal curvatures $\kappa_1$ and $\kappa_2$ . The principal directions

$$r_{max} = \text{arc}\tan(\lambda_1) \tag{53}$$

$$r_{min} = \text{arc}\tan(\lambda_2) \tag{54}$$

## 2.8 The Curvature of theGeneral surfaces with Cartesian Coordinates: Case Study Ellipsoid

First, we need to find the azimuth angle between the two points known as the Cartesian coordinates. Let's assume that θ is between azimuth angle of $P_o$ and $P_1$ points. For this we need the $P_o$ geodetic coordinates $(\Phi,\lambda)$. We may obtain the $P_o$ geodetic coordinates $(\Phi,\lambda)$ from its Cartesian coordinates $(x_o,y_o,z_o)$. Formulas related the geodetic and Cartesian coordinates conversion on a triaxial ellipsoid were expressed on Feltens (2009), Ligas (2012) andBektas (2014). For detailed information on this subject please refer to Bektas (2014). The following link can be used for the conversion of Cartesian coordinates to geodetic coordinates on triaxial ellipsoid (URL-1).

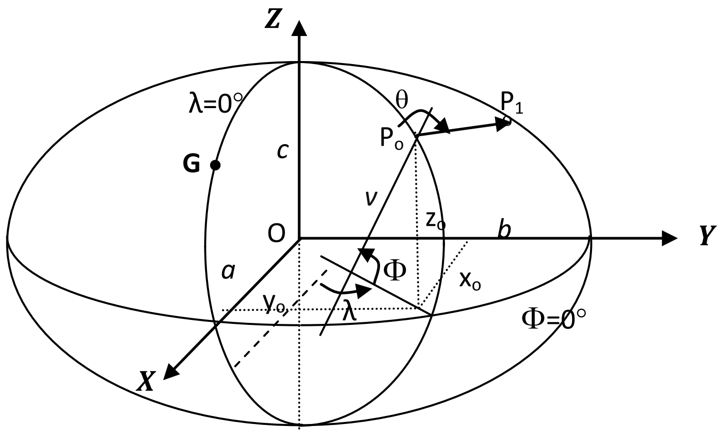


*Figure-4 (X,Y,Z) Cartesian and (Φ, λ, h) Geodetic coordinates onTriaxial ellipsoid*

$\Delta x = x_1 - x_0$ $\Delta y = y_1 - y_0$ $\Delta z = z_1 - z_0$

The azimuth angle of $(P_o-P_1)$ from Cartesian coordinates. Moritz,1980

$$\theta = (\mathbf{P_o}\text{-}\mathbf{P_1}) = \text{arc}\tan\left(\frac{-\Delta x\sin\lambda + \Delta y\cos\lambda}{-\Delta x\sin\Phi\cos\lambda - \Delta y\sin\lambda\sin\Phi + \Delta z\cos\Phi}\right) \tag{55}$$

$P_o$ geodetic coordinates $(\Phi,\lambda)$ calculated from its $(x_o,y_o,z_o)$ Cartesian coordinates (URL-1).

In order to calculation for the curvature, we need to add the reduction of the direction of $r_{\min}$ to the angle θ

Let's assume that $r$ is the reduction of the direction of minimum curvature $r_{min}$

$$r = \arctan\left(\frac{W\tan(r_{\min})}{E + F\tan(r_{\min})}\right) \tag{56}$$

For spheroid $r_{\min}$ and $r$ becomes zero

And we give a newformula for the curvature calculation depending on the θangle of the azimuth on the triaxial ellipsoid

$$\kappa_n(\theta) = \kappa_1 + (\kappa_2 - \kappa_1)\cos^2(\theta - r) \qquad \text{Bektas Formula} \tag{57}$$

Curvature calculation can also be made as follows:First a plane's equationis determined which contains$P_o$ surface of normal and passes $P_1$point. And then we find elliptical equation the intersection of the plane and the ellipsoidGrey 1997, Klein 2012,Ferguson 1979, URL-2. The curvature can be calculated at point $P_o$on the elliptical equation.

### 2.9 Comparison Euler formula and Bektas formula

Euler formula (Eq.43) does not apply in general surface.Euler's formula is only used for the rotational ellipsoid surface in curvature calculations. To use the Euler formula the surface must be orthogonal (F=0). In other words, the surfaces parameters ($u,v$) must be orthogonal and the main curvature should be on $u$, $v$ axis direction.Euler's formula is not used in the general situation.Bektas formula (Eq.57) is a generalized version of Euler's formula and it applicable to the general surfaces (e.g. general ellipsoid, paraboloid and hyperboloid). To use the formula we need to know the main curvature ($\kappa_1$, $\kappa_2$) and principal direction($r_{\min}$). Bektas formula is used easily curvature calculation with Cartesian coordinates.

## 3. Conclusion

This study aims to show how to obtain the curvature of the ellipsoid depending on azimuth angle. We have developed an algorithmto obtain the curvature of the ellipsoid depending on azimuth angle. The efficiency of the new approaches is demonstrated through a numerical example. The presented algorithm can be applied easily for spheroid, sphere and also other quadratic surface, such as paraboloid and hyperboloid. Today, backward and forward problem between the two points on the triaxial ellipsoid with geodetic coordinates could not be a clear solution. Our future work will be on this unsolvable problem. I hope, the result of this study will contribute to the solution of the above problem.

## 4. Numerical Example

Find the curvature of normal section curve at $P_o$ point which contains $P_o$ surface of normal and passes $P_1$ point on a triaxial ellipsoid

θAngle is a azimuth angle $P_o$-$P_1$ direction and Cartesian coordinatesof $P_o$ and $P_1$point are given below

$$\frac{x^2}{25} + \frac{y^2}{16} + \frac{z^2}{9} - 1 = 0 \qquad \text{(Ellipsoid equation)}$$

(a=5,b=4,c=3) semi-axis

Cartesian coordinates ($x,y,z$)

$P_o$(3.000 2.5001.4981) and $P_1$( 2.6189 2.41251.8047)

Geodetic coordinates (URL-1)

$\Phi_o$= 40.194814370 $^o$ $\lambda_o$= 52.47573738$^o$ $h_o = 0$

*u,v* surface parameters on $P_o$ point Eq.(4)

*u*= 46.1691393 *v*= 60.0413669

E, F, G, e, f, gare the Fundamental Coefficients of the First and Second Order evaluated at P point.Eqs.(5-12)

E= 15.52562500 F= -1.94530956 G= 11.82202800

e=2.91030887 f= 0 g= 3.87718085

Gaussian Curvature, Mean Curvature

From fundamental coefficients of the first and second order.

K- Gauss= 0.06277139 H-Mean Curve= 0.26313232 (Eqs.17-18)

Main radii of curvatures

$$R_1 = \frac{1}{\kappa_1} = \frac{1}{H - \sqrt{H^2 - K}}$$ = 5.47305756 maximum radii of curvature (Eq.21)

$$R_2 = \frac{1}{\kappa_2} = \frac{1}{H + \sqrt{H^2 - K}}$$ = 2.9107724 minimum radii of curvature Eq.(22)

Principal Curvatures
$\kappa_1$ = 0.182713 minimum curvature

$\kappa_2$ = 0.343551 maximum curvature

Principal directionsEq.(52)

$-7.54231697\,\lambda^2 + 25.7899\,\lambda + 5.66145 = 0$

$\lambda_1$ =3.62635252 $\lambda_2$ = -0.20699173

$r_{max}$ =arc tan($\lambda_1$) =74.583317 $^o$ $r_{min}$=arc tan($\lambda_2$)= -11.694599 $^o$

Azimuth angle ($P_o$-$P_1$) from Eq.(55)
θ =**($P_o$-$P_1$)** =30.136633 $^o$

$$r = \arctan\left(\frac{W\tan(r_{\min})}{E + F\tan(r_{\min})}\right)$$ ➔ $r = -9.88360428^o$

$\theta - r$ =30.136633$^o$– (-9.88360428$^o$) = 40.02023728$^o$

and curvature

$$\kappa_n(\theta) = \kappa_1 + (\kappa_2 - \kappa_1)\cos^2(\theta - r)$$ = 0.277041

**Control**

Equation of planewhich contains $P_o$ surface of normal and passes $P_1$ point

$0.3125x \ -0.50134y + \ 0.245316z \ -0.051655 = 0$

Intersection ellipse's equation URL- 2

$$\frac{x^2}{\eta^2} + \frac{y^2}{\xi^2} = 1 \qquad ==> y = \xi\sqrt{1 - \frac{x^2}{\eta^2}}$$

$\eta = 4.65998$ $\xi = 3.11078$

Transformed coordinates of $P_o$point in intersection's plane

$x_o = -3.7331 \ \ y_o = -1.8619$

$y_o' = 0.89347 \quad y_o'' = -0.66809$

and curvature

$$= \frac{1}{R} = \frac{y_o''}{(1+y_o'^2)^{3/2}} = 0.2770411$$

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