Some Results Related to Polynomial Identities

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Abstract **:** In this paper we have discussed about GroupRings and Polynomial Identities. We have proved the famous theorem of Amitsur and Levitzki on polynomial identities satisfied by K_m , the ring of all $m \times m$ matrices over K.

Introduction: *Let K be a field and G be a multiplicative* group. Let K[G] denote the set of all formal sums k=$\sum_{g\in G} k_g\, g$, where $k_g \in K$ for every g$\in G$ and the set $\{g \in G/k_g \neq 0\}$ is finite. For k=$\sum_{g\in G} k_g\, g$ and s=$\sum_{g\in G} s_g\, g$ belonging to K[G],define k+s= $\sum_{g\in G}(k_g + s_g)$. This defines addition in K[G] with respect to which K[G] becomes an abelian group. Again for $\alpha \in K$ and k=$\sum_{g\in G} k_g\, g \in K[G]$, we define $\alpha k = \sum_{g\in G}(\alpha k_g)g$. With respect to this scalar multiplication, K[G] is vectorspace over K. For k=$\sum_{g\in G} k_g\, g$ and s=$\sum_{g\in G} s_h\, h$,define $ks = \sum_{l\in G} t_l\, l$ where $t_l = \sum_{gh=l} k_g s_h$ and the elements of G commutes with the elements of K. With the multiplication as defined above, K[G]becomes a ring. Hence K[G] is an algebra over the field K and is called group ring over K.

For $\propto = \sum_{g\in G} k_g\, g$ we define $supp(\propto) = \left\{g \in \frac{G}{k_g} \neq 0\right\}$ and is called support of $\propto$ .Clearly $supp(\propto)$ is a finite subset of G. It is easy to see rhat if $\propto$ is central in K[G] and $x \in supp(\propto)$ then all the conjugates of x are in $supp(\propto)$ and therefore there are only a finite number of distinct x^y with $y \in G$.

Next we define polynomial identity. Let $K[X_1, X_2 \sum \quad , \ldots]$ be the polynomial ring over a field K in the noncommuting indeterminates $X_1, X_2, \ldots$. An algebra E over K is said to satisfy a polynomial identity,if there exists $f(X_1, X_2, \ldots, X_n) \in K[X_1, X_2, \ldots], f \neq 0\ \forall \alpha_1, \alpha_2, \ldots, \alpha_n \in E$.

For example,any commutative algebra satisfies $f(X_1, X_2) = X_1X_2 - X_2X_1$.

The standard polynomial of degree n is defined by

$[X_1, X_2, \ldots, X_n]$=$\sum_{\sigma\in S_n}(-1)^\sigma X_{\sigma(1)}, X_{\sigma(2)}, \ldots, X_{\sigma(n)}$.

Here S_n is the symmetric group of degree n on the set $S = \{1,2, \ldots, n\}$ and $(-1)^\sigma$ is 1 or -1 according as σ is an even or an odd permutation. We will also use $s_n(X_1, X_2, \ldots, X_n)$ to denote this polynomial. Now we require following lemmas to prove the result that K_m , the ring of all $m \times m$ matrices over a field K, satisfies the standard polynomial identity of degree $2m$.

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Lemma 1.1 . Suppose E is an algebra over a field K which satisfies a nontrivial polynomial identity of degree n. Then E satisfies the polynomial identity $f \in K[X_1, X_2, \dots, X_n]$ of the form

$f = \sum_{\sigma \in S_n} a_\sigma X_{\sigma(1)} X_{\sigma(2)} \dots X_{\sigma(n)}$ where $a_\sigma \in$ K and they are not all zero.

Lemma 1.2 . Let E=K_m be the ring of all $m \times m$ matrices over K. Then E does not satisfy a polynomial identity of degree less than $2m$.

Proof. Suppose by way of contradiction that E satisfy a polynomial identity of degree $n < 2m$. By lemma 1.1 , E satisfies $f = X_1X_2 \dots X_n + \sum_{\substack{\sigma \in S_n \\ \sigma \neq 1}} a_\sigma X_{\sigma(1)} X_{\sigma(2)} \dots X_{\sigma(n)}$.Let $\{e_{ij}\}$ denote the matrixunits in E i.e. e_{ij} is the matrix whose only nonzero entry is 1 in the (i,j)th position. Since $n < 2m$,we may set $X_1 = e_{11}, X_2 = e_{12}, X_3 = e_{22}, X_4 = e_{23}, X_5 = e_{33,\dots}$

Then $X_1X_2 \dots X_n \neq 0$ at these values as we know that $e_{ij}\, e_{kl} = 0$ if $j \neq k$ and e_{il} if $= k$.

But $X_{\sigma(1)} X_{\sigma(2)} \dots X_{\sigma(n)} = X_1X_2 \dots X_n \neq 0$ at these values. Thus E does not satisfy . We now proceed to show that K_m does infact satisfy a polynomial identity of degree $2m$, namely the standard polynomial identity. It is easy to see that for a positive integer $r, 1 \leq r \leq m, K_m^{(r)}$ (the set of all matrices in K_m whose rth row and rth coloumn are zero) is an algebra isomorphic to K_{m-1} .

Lemma 1.3. The standard polynomial $s_n(X_1, X_2, \dots, X_n)$ has the following properties.

i) s_n is linear in each variable.

ii) $s_n(\dots, X_j, \dots, X_i, \dots) = -s_n(\dots, X_i, \dots, X_j, \dots)$
Thus upto a $\pm$ sign, all ordering of the variables are equivalent.

iii) $s_n(\dots, X_i, \dots, X_i, \dots) = 0$.

iv) $s_{n+1}(X_1, X_2, \dots, X_{n+1}) = \sum_i \pm X_i s_n(X_1, \dots, \hat{X}_i, \dots, X_{n+1})$.
Thus if an algebra E satisfies s_n, then it satisfies s_{n+1}.

v) $s_{2n}(1, X_2, \dots, X_{2n}) = 0$.

Lemma 1.4.

i) Let $s_3$be the sum of all those terms in $s_n(X_1, X_2, \dots, X_n)$ in which the product $X_1X_2X_3$ occurs. Then $s_3 = s_{n-2}(X_1X_2X_3, X_{4,} \dots, X_n)$.

ii) Let $s_2$be the sum of all those terms in $s_n(X_1, X_2, \dots, X_n)$ in which the product X_1X_2 occurs. Then
$s_2 = \pm s_{n-2}(X_3, X_{4,} \dots, X_n) X_1X_2 + \sum_3^n \pm s_{n-2}(X_1X_2X_i, X_{3,} \dots, \hat{X}_i, \dots., X_n)$.

Lemma 1.5. Let $m > 1$ and assume that K_{m-1} satisfies the polynomial identity s_{2m-2} . Let $\{e_{a_ib_i}\}$ be a set of 2 m matrix units in $K_m$and suppose that $s_{2m(e_{a_1b_1}, e_{a_2b_2}, \dots, e_{a_{2m}b_{2m}}) \neq 0}$. For each

$u = 1,2, \dots, m$, let $w(u)$ equal the number of times u occurs as a subscript in $\{e_{a_i b_i}\}$ .Then the possible configuration for the values $w(u)$ in some order are

3,5,4,4,4,4,…

3,3,6,4,4,4,…

OR

4,4,4,4,4,4,…

Theorem 1.6. K_m, the ring of all $m \times m$ matrices over a field K, satisfies the standard polynomial identity of degree 2 m.

Proof. We proceed by induction on . The case $m = 1$ is clear since $K_1$is commutative and $s_2(X_1, X_2) = X_1X_2 - X_2X_1$.Let $m > 1$ and suppose that K_{m-1} satisfies s_{2m-2}.

Let e and f be two orthogonal primitive idempotent in K_m. We show first that $s_{2m}(\mathrm{e}, \mathrm{f}, K_m, K_m, \dots, K_m) = 0$. We can ofcourse choose a basis of matrix units so that e $= e_{11}$, f $= e_{22}$. If $s_{2m}(\mathrm{e}, \mathrm{f}, K_m, K_m, \dots, K_m) \neq 0$ then by linearity there exists a set of matrix units $\{e_{a_i b_i}\}$ containing e_{11} and e_{22} with $s_{2m}\left(e_{a_1 b_1}, e_{a_2 b_2}, \dots, e_{a_{2m} b_{2m}}\right) \neq 0$. By lemma 1.5 , there exists at most one u with $w(u) > 4$ and thus either $w(1) \leq 4$ or $w(2) \leq 4$ .We then replace e_{11} in $s_{2m}(. \ . \ .)$ by $e_{11} = 1 - \sum_{i=2}^{m} e_{ii}$ and use linearity. By lemma1.3 (v), $s_{2m}(1, . \ . \ .) = 0$. In all the other terms , we have $w(1) \leq 4 - 2 = 2$,so these terms must also vanish by lemma 1.5, a contradiction.

Let e be a primitive idempotent in K_m . We show now that $s_{2m}(\mathrm{e}, K_m, K_m, \dots, K_m) = 0$. Choose a basis of matrix units so that e $= e_{11}$ and consider $s_{2m}\left(e_{a_1 b_1}, e_{a_2 b_2}, \dots, e_{a_{2m} b_{2m}}\right)$ with $e_{11} \in \{e_{a_i b_i}\}$. If all $e_{a_i b_i}$ have a subscript 1, then since there are at the most $2m - 1$ such matrix units it follows that two entries in $s_{2m}(. \ . \ .)$ are equal and hence $s_{2m}(. \ . \ .) = 0$ by lemma 1.4(iii).

(Let r be the number of matrix units of the form e_{1i}, $1 \leq i \leq m$ and s be the number of matrix units of the form e_{j1}, $1 \leq j \leq m$. Then r + s $\geq 2m - 1$. We claim that either r $>$ m or s $> m$. Suppose neither r $> m$ or s $> m$. Then r $\leq m - 1$, s $\leq m - 1$ and so r + s $\leq 2m - 2$, a contradiction. Suppose r $> m$. Since $1 \leq i \leq m$ therefore there exists some i such that e_{1i} is repeated. Thus two entries in $s_{2m}(. \ . \ .)$ are equal .).

If not then some e_{ij} occurs with $i, j \neq 1$. By preceeding paragraph and using 1.3(ii), $s_{2m}(. \ . \ .) = 0$ if $i = j$, so assume that $i \neq j$. Then $e_{ij} = e_{ii} + e_{ij} - e_{ii}$ is diference of two primitive idempotents each orthogonal to e_{11}. By linearity and the result of preceeding paragraph , we have $s_{2m}(. . .) = 0$ using 1.3(ii).

Finally we show that $s_{2m}(K_m, K_m, ..., K_m) = 0$. By linearity , it is sufficient to show that $s_{2m}\left(e_{a_1 b_1}, e_{a_2 b_2}, ..., e_{a_{2m} b_{2m}}\right) = 0$. Let $e_{ab} \in \{e_{a_i b_i}\}$. If $a = b$ then e_{aa} is a primitive idempotent and $s_{2m}(. . .) = 0$ by the above . If $a \neq b$ then $e_{ab} = (e_{aa} + e_{ab} - e_{aa})$ is difference of two primitive idempotents and hence by linearity and above we have $s_{2m}(. . .) = 0$ using lemma 1.3(ii) . Thus K_m satisfies s_{2m} and the result follows by induction.

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