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Orthogonal (Shortest) Distance To the Hyperboloid

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ABSTRACT

The hyperboloid of one sheet is possibly the most complicated of all the quadric surfaces. For one thing, its equation is very similar to that of a hyperboloid of two sheets, which is confusing. For another, its cross sections are quite complex. Hyperboloid has a complex geometry as well as hyperboloid structures have always been interested. The two main reasons, apart from aesthetic considerations, are strength and efficiency.

Finding the orthogonal (shortest) distance from a point to a Hyperboloid surface corresponds to the normal height in Geometry. The problem of finding the shortest distance is encountered frequently in various optimization problem, especially best fitting hyperboloid problem.

Keywords:

Hyperboloid; orthogonal distance, 3D reverse transformation;hyperboloid structures**,** Best Fitting Hyperboloid

1. Introduction

As hyperboloid structures are double curved, that is simultaneously curved in opposite directions, they are very resistant to buckling. This means that you can get away with far less material than you would otherwise need, making them very economical. Single curved surfaces, for example cylinders, have strengths but also weaknesses.

Take a drinks can for example: these are made extremely thin, with sides only a fraction of a millimetre thick, yet contain the pressurised beverage and if stood on end can support the weight of a grown adult even when empty. But, once you have enjoyed the contents, you can push in the side with just a slight pressure from your finger. Alternately, if you were to push with your finger from the inside of the can (being careful to avoid any sharp edges of course) then you will find that you have to apply considerable effort to make any impression.

Double curved surfaces, like the hyperboloids in question, are curved in two directions and thus avoid these weak directions. This means that you can get away with far less material to carry a load, which makes them very economical.

The second reason, and this is the magical part, is that despite the surface being curved in two directions, it is made entirely of straight lines. Apart from the cost savings of avoiding curved beams or shuttering, they are far more resistant to buckling because the individual elements are straight Andrews and Séquin(2013), Mollin (1995).

This is an interesting paradox: you get the best local buckling resistance because the beams are straight and the best overall buckling resistance because the surface is double curved. Hyperboloid structures cunningly combine the contradictory requirements into one form. The hyperboloid is the design standard for all nuclear cooling towers and some coal-fired power plants. It is structurally sound and can be built with straight steel beams.

When designing these cooling towers, engineers are faced with two problems:
(1) the structure must be able to withstand high winds and
(2) they should be built with as little material as possible.
The hyperbolic form solves both of these problems. For a given diameter and height of a tower and a given strength, this shape requires less material than any other form URL-1.
The pioneer of hyperboloidal structures is the remarkable Russian Engineer V. Shukhov (1853-1939) who, among other accomplishments, built a hyperboloidal water tower for the 1896 industrial exhibition in Nizhny Novgorod. Hyperboloidal towers can be built from reinforced concrete or as a steel lattice, and is the most economical such structure for a given diameter and height. The roof of the McDonnell Plantarium in St. Louis, the Brasilia Cathedral and the Kobe Port tower are a few recent examples of hyperboloidal structures. The most familiar use, however, is in cooling towers used to cool the water used for the condensers of a steam power plant, whether fuel burning or nuclear. The bottom of the tower is open, while the hot water to be cooled is sprayed on wooden baffles inside the tower. Potentially, the water can be cooled to the wet bulb temperature of the admitted air. Natural convection is established due to the heat added to the tower by the hot water. If the air is already of moderate humidity when admitted, a vapor plume is usually emitted from the top of the tower. The ignorant often call this plume "smoke" but it is nothing but water. Smokestacks are the high, thin columns emitting at most a slight haze. The hyperboloidal cooling towers have nothing to do with combustion or nuclear materials. Two such towers can be seen at the Springfield Nuclear Plant on The Simpsons. The large coal facility at Didcot, UK also has hyperboloidal cooling towers easily visible to the north of the railway west of the station. Hyperboloidal towers of lattice construction have the great advantage that the steel columns are straight.URL-2

A cooling tower

First, the basic definition of Hyperboloid starts with giving mathematical equations to explain the concepts. To show how computations of the shortest distance to an Hyperboloid, are carried out, we solve this problem separately: standard Hyperboloid and the shifted-oriented Hyperboloid. The efficacy of the new algorithms is demonstrated through simulations Hilbert and Cohn-Vossen (1999).
When we look at literature in this regard, we often see studies about ellipsoidal distances. We can develop a distance finding algorithm for hyperboloid by simulating it. And we did so, but this process is a little more difficult than the ellipsoid. In hyperboloids, one or two semi-axes are negative, such as the change of the order of half-axis size.
In the literature on this subject we see the various studies :Eberly (2008), Feltens (2009) , Ligas (2012) and Bektas(2014). For the solution Eberly (2008) gives a method that is based on sixth degree polynominal. He has benefited from the largest root of 6th degree polynomial. Feltens (2009) gives a vector-based iteration process for finding the point on the ellipsoid. Ligas (2012) claims his method turns out to be more accurate, faster and applicable than Feltens method. Bektas(2014) is a little more advanced than the Ligas (2012), and autor gives a Matlab code for this calculation.
The presented paper tries to give the shortest distance calculation not only for the hyperboloid in standard position but also the shifted-oriented hyperboloid. We could not find enough studies with numerical examples on this subject in the literature, especially for shifted-oriented hyperboloid.

2. Definition of Hyperboloid

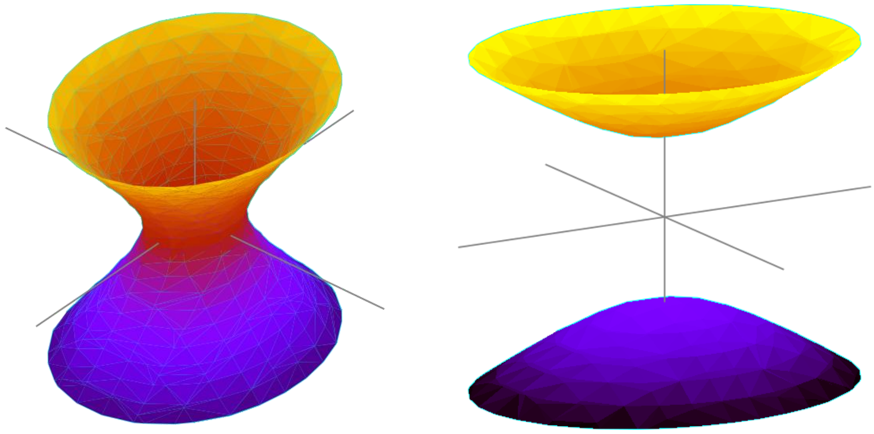

Figure-1 Hyperboloid

a)Hyperboloid of one sheet b)Hyperboloid of two sheets

Hyperboloid a geometric surface consisting of one sheet, or of two sheets separated by a finitedis tance whose sections paralleltothethreecoordinateplanesarehyperbolasor ellipses. The standard equation of a hyperboloid centered at the origin of a cartesian coordinate system and aligned with the axes is given below.

Let a hyperboloid be given with the three semi axes a, b, c see Fig.1

$$\frac{X^2}{a^2}+\frac{Y^2}{b^2}-\frac{Z^2}{c^2}=\pm 1 \quad \text{(Hyperboloid equation)} \qquad (1)$$

+1 where on the right hand side of (1) corresponds to a hyperboloid of one sheet, on the right hand side of -1 to a hyperboloid of two sheets.

2.1 Generalised Equation of Hyperboloid

More generally, an arbitrarily oriented hyperboloid, centered at **v**, is defined by the equation

$$(\mathbf{x}-\mathbf{v})^T A (\mathbf{x}-\mathbf{v}) = 1 \qquad (2)$$

where *A* is a matrix and **x**, **v** are vectors.

The eigenvectors of *A* define the principal directions of the hyperboloid and the eigenvalues of A are the reciprocals of the squares of the semi-axes: $1/a^2$, $1/b^2$ and $1/c^2$. The one-sheet hyperboloid has two positive eigenvalues and one negative eigenvalue. The two-sheet hyperboloid has one positive eigenvalue and two negative eigenvalues. We will continue our studies on hyperboloid of one sheet.

3. Finding The Point On The Hyperboloid

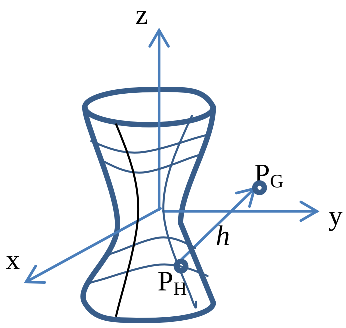

Figure 2-Standard Hyperboloid

In this section, we assume that the ellipsoid is in standard position, in other words its axis is aligned and centered at the origin (Figure -1). Findingthe shortest distance to a hyperboloid is corresponds the hyperboloid height in geometry. The most time-consuming part is the computation of the orthogonal distance between each point and the hyperboloid.To find the projection of an external point denoted as P_G (x_G,y_G,z_G) in (Fig.1) above you can see that the hyperboloid height is along the normal to this surface i.e. point P_H (x_H,y_H, z_H) . Here, there are two different situations: one of is computation on *xyz* system (standard hyperboloid position), the other is computations on *XYZ* system (shifted-oriented hyperboloid position). Firstly, we assume that the hyperboloid is in standard position, in other words its axis is aligned and centered at the origin. Secondly we will discuss the other hyperboloid.

There are two methods for solving this problem, one being direct other iterative.

3.1-Direct method

The distance formula between P_G and P_H

$$h = P_H P_G = \sqrt{(x_H - x_G)^2 + (y_H - y_G)^2 + (z_H - z_G)^2} \quad (3)$$

On the other hand the P_H point should be on the hyperboloid

$$\frac{X_H^{\ 2}}{a^2} + \frac{Y_H^{\ 2}}{b^2} - \frac{Z_H^{\ 2}}{c^2} = 1 \quad (4)$$

If we pull x_H out of the equation 2 and put it in the equation 1 we obtain the following equation.

$$f = (y_H - y_G)^2 + (z_H - z_G)^2 + \left(x - a\sqrt{1 - (\frac{z_H}{c})^2 - (\frac{y_H}{b})^2} \right)^2 \quad (5)$$

The partial derivatives of the equation are equal to zero so that the distance is minimum

$$f_z = \frac{\partial f}{\partial z} = 2(z_H - z_G) + \frac{2.a.z_H \left(x - a\sqrt{1-(\frac{z_H}{c})^2 - (\frac{y_H}{b})^2} \right)}{c^2 \sqrt{1-(\frac{z_H}{c})^2 - (\frac{y_H}{b})^2}} = 0 \quad (6)$$

$$f_y = \frac{\partial f}{\partial y} = 2(y_H - y_G) + \frac{2.a.y_H \left(x - a\sqrt{1-(\frac{z_H}{c})^2 - (\frac{y_H}{b})^2} \right)}{b^2 \sqrt{1-(\frac{z_H}{c})^2 - (\frac{y_H}{b})^2}} = 0 \quad (7)$$

The common solution of these two equations will give the y_H and z_H coordinates of P_H point. We substitute these coordinates in the hyperboloid equation to obtain x_H values.

3.2-Iteratif method

It can be proved that the shortest distance is along the surface normal. The first step is to find the projection of an external point denoted as $P_G(x_G,y_G,z_G)$ as shown in Figure-2 on this hyperboloid along the normal to this surface i.e. point $P_H(x_H,y_H,z_H)$.

h = $P_H P_G$: normal distance, the shortest distance, orthogonal distance

The following link can be used for projection coordinates and shortest distance on hyperboloid or triaxial ellipsoid URL-3.

http://www.mathworks.com/matlabcentral/fileexchange/46261-the-shortest-distance-from-a-point-to-ellipsoid

According to Bektas(2014) we can get the solution from three nonlinear equations. These equation are two of them a collinearity condition can be written between P_H and P_G and one of them the coordinates of P_H must satisfy the equation of the hyperboloid. For detailed information see Bektas(2014).

$$\frac{x_H - x_G}{E.x_H} = \frac{y_H - y_G}{F.y_H} = \frac{z_H - z_G}{G.z_H} \quad (8)$$

These three equations are linearized by Taylor series expansion and the system of equations is solved in order to obtain the solution for $X_H = [x_H, y_H, z_H]$.

$j_{11}=F.y_H-(y_H-y).E$; *$j_{12}=(x_H-x).F-E.x_H$;* *$j_{13}=0$;*

$j_{21}=G.z_H-(z_H-z).E$; *$j_{22}=0$;* *$j_{23}=(x_H-x).G-E.x_H$;*

$j_{31}=2.E.x_H$ *$j_{32}=2F.y_H$* *$j_{33}=2G.z_H$*

$$A = \begin{bmatrix} j_{11} & j_{12} & 0 \\ j_{21} & 0 & j_{23} \\ j_{31} & j_{32} & j_{33} \end{bmatrix} \qquad \delta_H = \begin{bmatrix} \delta x_H \\ \delta y_H \\ \delta z_H \end{bmatrix} \qquad f = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} \quad (9)$$

$A\,\delta_{i} + f = 0$ (10)

can be solved very easily in MATLAB

$\delta_i = - A \backslash f$ (11)

The entries may be written as follows

Replacing x_G ,y_G , z_G for respectively x , y , z in the belov equations , we find:

$f1 = (x_H - x).\,F.\,y_H - (y_H - y).E.x_H$ (12.a)

$f2 = (x_H - x).\,G.z_H - (z_H - z).E.x_H$ (12.b)

$f3 = E.x_H^2 + F.y_H^2 + G.z_H^2 - 1$ (12.c)

$$E = \frac{sign(a)}{a^2} \qquad F = \frac{sign(b)}{b^2} \qquad G = \frac{sign(c)}{c^2} \qquad (13)$$

Case : Shifted-oriented hyperboloid

In generally, for shifted-oriented hyperboloid as in (Fig.3), the data point $P_G(X_G, Y_G, Z_G)$ can be rotated and translated to axis-aligned hyperboloid centered at the origin and the distances can be calculated in that system. For this conversion we utilize performed as follows by making use of hyperboloid 's center coordinates (X_o, Y_o, Z_o) and the rotation angles (ε, ψ, ω) , in accordance with (Fig.3). If we do not know the hyperboloid parameters but we know the conic equation of hyperboloid as below.

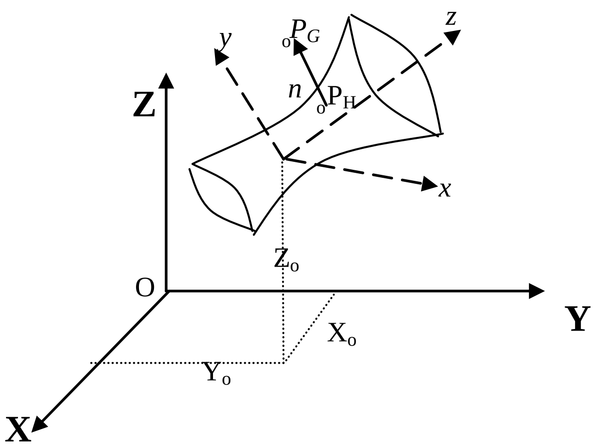

Figure3-Shifted-oriented hyperboloid

$A\,x^2 + B\,y^2 + C\,z^2 + 2D\,xy + 2E\,xz + 2F\,yz + 2G\,x + 2H\,y + 2I\,z - 1 = 0$ (14)

We can reach these parameters from this conic equation. To obtain hyperboloid parameter from the conic equation see Bektas(2014), Bektas(2015). The center coordinates of hyperboloid and the angles of rotation are needed to make this transformation. The shortest distance (h) calculation is made of the new converted coordinates $P_G(x_G, y_G, z_G)$ in standard position with the above procedure and the coordinates of $P_H\,(x_H, y_H, z_H)$ in standard position are found.

$$\begin{bmatrix} x_G \\ y_G \\ z_G \end{bmatrix} = inv(R) \begin{bmatrix} X_G - X_o \\ Y_G - Y_o \\ Z_G - Z_o \end{bmatrix} \quad (15)$$

To find the true coordinates of $P_H(x_H, y_H, z_H)$ we need to make a transformation as below:

$$\begin{bmatrix} X_H \\ Y_H \\ Z_H \end{bmatrix}_{True} = \begin{bmatrix} X_o \\ Y_o \\ Z_o \end{bmatrix} + R \begin{bmatrix} x_H \\ y_H \\ z_H \end{bmatrix}_{Standart} \quad (16)$$

The following link can be used for hyperboloid parameter from the conic equation.
http://www.mathworks.com/matlabcentral/fileexchange/48974-conversion-from-conic-parameters-to-geometric-parameters-of-hyperboloids/content/Conic_EllipsoidParameter.m

4. Obtaining Hyperboloid Parameters from Conic Equation

This section we determines the center, semi-axis and rotation angles of the hyperboloid. We first need to know the hyperboloid conic equation for this problem. The general conic equation of an hyperboloid is given as Eq.14

$A x^2 + B y^2 + C z^2 + 2D\,xy + 2E\,xz + 2F\,yz + 2G\,x + 2H\,y + 2I\,z - 1 = 0$

$$\begin{bmatrix} A & D & E \\ D & B & F \\ E & F & C \end{bmatrix} \begin{bmatrix} X_o \\ Y_o \\ Z_o \end{bmatrix} + \begin{bmatrix} G \\ H \\ I \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (17)$$

The solution of above equation system which is established conic coefficients gives us the coordinates of hyperboloid's center (X_o, Y_o, Z_o).

For finding of semi-axis (a,b,c) and rotation angles of the hyperboloid $(\varepsilon, \psi, \omega)$:
Firstly eigenvalues and eigenvectors of above coefficient matrix (S_{3x3}) in Eq.(17) can be easily calculated with the following MATLAB command

[evecs, evals]=eig (S) (18)

evals : Eigenvalues of (S) $=[\lambda_1 \;\; \lambda_2 \lambda_3]^T$

Let λ_1, λ_2 and λ_3 get the eigenvalues of the matrix S, in descending order Semi-axes of

hyperboloid (a,b,c) obtained the eigenvalues of S as belov

a = $sign(\lambda_1)/\sqrt{abs(\lambda_1)}$ b= $sign(\lambda_2)/\sqrt{abs(\lambda_2)}$ c = $sign(\lambda_3)/\sqrt{abs(\lambda_3)}$ (19)

It should not be forgotten that The one-sheet hyperboloid has two positive eigenvalues and one negative eigenvalue. The two-sheet hyperboloid has one positive eigenvalue and two negative eigenvalues. So it is necessary to pay attention to this in the rooting process. Rooting can be done as above.

evecs : Eigenvectors of (S) give us R - rotatiton angles of hyperboloid

5. Numeric Example-1(for standard hyperboloid)

In order to demonstrate the validity of the shortest distance algorithms presented above, a numerical example is given. The algorithm was implemented in MATLAB.

$$\frac{X^2}{25}+\frac{Y^2}{16}-\frac{Z^2}{9}=1$$

semi axes a=5, b=4, c=3

Outside point coordinates $x_G = 6$, $y_G = 5$, $z_G = 4$.

Finding the shortest distance to hyperboloid.

For this, we must firstly find P_H point Cartesian coordinates. The initial guesses for the P_H point on the hyperboloid we can get the coordinates of the P_G point.

The following link can be used for projection coordinates and shortest distance on hyperboloid or triaxial ellipsoid URL-3.

http://www.mathworks.com/matlabcentral/fileexchange/46261-the-shortest-distance-from-a-point-to-ellipsoid

[PH,dis] = shortest_distance([6 5 4],[5 4 -3])

P_H = (5.926, 4.904, 4.1438) ➔ P_H (x_H, y_H, z_H)

dis = 0.1879 ➔h = $P_H P_G$: normal distance, the shortest distance, orthogonal distance.

Numeric Example-2 (shifted-oriented hyperboloid)

A shifted-oriented hyperboloid equation is given as

$A x^2 + B y^2 + C z^2 + 2D\, xy + 2E\, xz + 2F\, yz + 2G\, x + 2H\, y + 2I\, z - 1 = 0$

Coefficients of hyperboloid equation as below

[0.001049 -0.0014285 0.000994 -0.0047292 0.0106161 -0.0062316 -0.244882 0.291385 -0.041201 -1]

The coordinates of outside point P_G

x_G =17.405 y_G =21.0227 z_G = 34.5953

Finding the shortest distance to hyperboloid.

hyperboloid 's center coordinates (X_o, Y_o, Z_o) = (10, 20 , 30) from Eq.17

hyperboloid 's radii = (5, 4 , -3) from Eq.18-19

R - Rotation matris of hyperboloid from Eq.18

$$R = \begin{bmatrix} -0.642787 & 0.3830 & -0.66341 \\ 0.492403 & 0.8700 & 0.02520 \\ -0.586824 & 0.3104 & 0.74782 \end{bmatrix}$$

P_H = (17.0543, 24.4378, 32.4847) ➔ P_H (x_H, y_H, z_H)

dis = 0.1879 ➔ $h = P_H P_G$: normal distance, the shortest distance, orthogonal distance

6. Conclusion

In this paper we study on the computation the shortest distance from a point to a hyperboloid. The problem offinding the shortest distanceproblemsareencounteredfrequently inthefitting hyperboloid, image processing, face recognition, computer games etc. The paper has presented a new method of shortest distance to a hyperboloid. The new method relies on solving a overdetermined system of nonlinear equations with the use of a generalized Newton method. It has been compared to the other existing methods. In conclusion, the presented method may be considered as fast, accurate and reliable and may be successfully used in other areas. The presented algorithm can be applied easily for sphere, triaxial ellipsoid and also other surface such as paraboloid.

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