SOME *PROBLEMS IN GROUP RINGS*

Preeti Panwar
Assistant Professor in Mathematics
Guru Nanak Khalsa College Karnal

Abstract: *In this paper we prove that group ring K[G] is semiprime if ch(K) is zero.*

Introduction: *Let K be a field and G be a multiplicative* group. Let K[G] denote the set of all formal sums k=$\sum_{g\in G} k_g\, g$, where $k_g \in K$ for every g$\in G$ and the set $\{g \in G/k_g \neq 0\}$ is finite. For k=$\sum_{g\in G} k_g\, g$ and s=$\sum_{g\in G} s_g\, g$ belonging to K[G],define k+s= $\sum_{g\in G}(k_g + s_g)$. This defines addition in K[G] with respect to which K[G] becomes an abelian group. Again for $\alpha \in K$ and k=$\sum_{g\in G} k_g\, g \in K[G]$, we define $\alpha k = \sum_{g\in G}(\alpha k_g)g$. With respect to this scalar multiplication, K[G] is vectorspace over K. For k=$\sum_{g\in G} k_g\, g$ and s=$\sum_{g\in G} s_h\, h$,define $ks = \sum_{l\in G} t_l\, l$ where $t_l = \sum_{gh=l} k_g s_h$ and the elements of G commutes with the elements of K. With the multiplication as defined above, K[G]becomes a ring. Hence K[G] is an algebra over the field K and is called group ring over K.

For $\propto = \sum_{g\in G} k_g\, g$ we define $supp(\propto) = \left\{g \in \frac{G}{k_g} \neq 0\right\}$ and is called support of $\propto$.

Clearly $supp(\propto)$ is a finite subset of G. It is easy to see rhat if $\propto$ is central in K[G] and $x \in supp(\propto)$ then all the conjugates of x are in $supp(\propto)$ and therefore there are only a finite number of distinct x^y with $y \in G$.

Finite Conjugate Group **:** Let G be a group. We define $\Delta = \Delta(G) = \{x \in G/[G: C_G(x)] < \infty\}$ where $C(x) = C_G(x)$ is the centralizer of x in G. Let $\in \Delta$. Since there is one one and onto correspondence between the conjugates of x and the right cosets of $C(x)$ in G , it follows that x has finitely many conjugates iff number of distinct right cosets of $C(x)$ in G is finite *i.e.*$[G: C_G(x)] <\infty$ *i.e.* iff $x \in \Delta$. Thus $x \in \Delta$ iff x has finitely many conjugates. It is easy to see that Δ is characteristic subgroup of G and is called *finite conjugate subgroup* of G.

Consider the group ring K[G] over a field K. Let $\Delta = \Delta(G)$be the finite conjugate subgroup of G. Consider the subring K[Δ]={$\propto \in K[G]/supp(\propto)$} of $K[G]$. We define a map $\theta : K[G] \to K[\Delta]$ as follows:

Let $\propto = \sum_{x\in G} k_x x \in K[G]\ where\ k_x \in K, x \in G$ then $\propto = \sum_{x\in\Delta} k_x x + \sum_{x\notin\Delta} k_x x$. We define $\theta(\propto) = \sum_{x\in\Delta} k_x\, x$. Clearly θ is a K −linear map . Next we prove that θ is not a ring homomorphism. Let $x \in G$ be such that x has infinitely many conjugates but then $\notin \Delta \Rightarrow$ $\theta(x) = 0$. Since x and x^{-1} have same number of conjugates therefore $(x^{-1}) = 0$.

Now $(1) = \theta(xx^{-1}) = \theta(x)\theta(x^{-1}) = 0$. But $\theta(1) = 1$ as $1 \in \Delta$. Hence θ is not a ring homomorphism. Clearly , $Supp(\propto_1) \cap \Delta = \emptyset\ where\ \propto = \theta(\alpha) + \alpha_1$.

Theorem 1.1 : Suppose that K is a subfield of complex numbers which is closed under complex conjugation. Then K[G] contains no nonzero nil ideal.

Proof : Let * denotes complex conjugation and extend it to a map of K[G] to itself by $\alpha = \sum_{x\in G} k_x x \to \alpha^* = \sum_{x\epsilon G} k_x^* x^{-1}$

Clearly $(\alpha^*)^* = \alpha \; and \; (\alpha\beta)^* = \beta^*\alpha^*$. Now $\alpha\alpha^* = \sum_{x\in G} |k_x|^2$ for every $= \sum_{x\in G} k_x x \in K[G]$.

Thus coefficient of $1 \in G$ in $\alpha\alpha^*$is $\sum_{x\in G} |k_x|^2$ and therefore $\alpha\alpha^* = 0$ iff $\sum_{x\in G} |k_x|^2 = 0$ i.e. iff $k_x = 0 \; \forall \; x \in G$. Hence $\alpha\alpha^* = 0$ iff $\alpha = 0$.

Let I be a nil ideal in K[G] and $0 \neq \alpha \in I$. Since I is an ideal in K[G] and $\alpha^* \in K[G]$ we have $\alpha\alpha^* \in I$ and hence there exists some $n \geq 1$ such that $(\alpha\alpha^*)^n = 0$. Let n be minimal with this property. Suppose that $n > 1$ and let $= (\alpha\alpha^*)^{n-1}$. Clearly $\beta = \beta^*$ so

We have $\beta\beta^* = (\alpha\alpha^*)^{2n-2}$. Since $n > 1$ therefore $2n - 2 > n$ and hence $\beta^* = 0$. Thus $\beta = 0$, contradicting the minimality of . This shows that $= 1$. Hence $\alpha\alpha^* = 0$ and therefore $\alpha = 0$, a contradiction. Hence $I = 0$ and the result follows.

Theorem 1.2 : Let K be a field of characteristic 0. Then K[G] is semiprime.

Proof : Suppose K[G] is not semiprime. Then K[G] contains a nonzero ideal *A* with $A^2 = 0$. Let $\alpha = \sum_{i=1}^{n} k_i x_i \in A$ be such that $\alpha \neq 0$. Let F be a subfield of K generated over the rationals by $k_1 , k_2 , \ldots , k_n$. Then $F[G] \subseteq K[G] (\text{since ch(K)} = 0 \text{ , therefore Q} \subseteq \text{K})$ and $A \cap F[G]$ is a nonzero ideal in $F[G]$ of square zero. Therefore, it is clearly sufficient to assume that $K = F$ or equivalently that $K = F$ or equivalently that K is finitely generated over the rationals. Thus $\subseteq$ ¢ , the field of complex numbers and hence $K[G] \subseteq$ ¢$[G]$ and A¢ is a nonzero ideal of ¢$[G]$ with square zero. A contradiction to the above theorem and the result follows.

We now consider the field of characteristic p>0. Let R be a ring. We set [R,R] equal to the set of all finite sums of Lie products $[\alpha, \beta] = \alpha\beta - \beta\alpha$ with $\alpha, \beta \in R$ (i.e. $[R, R]$ is the additive subgroup of R generated by all the Lie products $[\alpha, \beta], \alpha, \beta \in R$).We will use following lemmas to prove next theorem.

Lemma 1.3.1 : Let E be an algebra over a fiekd K of characteristic p>0. Let k and n be positive integers. If $\alpha_1, \alpha_2, \ldots, \alpha_n \in E$ then

$$(\alpha_1 + \alpha_2 + \cdots + \alpha_n)^{p^k} = {\alpha_1}^{p^k} + {\alpha_2}^{p^k} + \cdots + {\alpha_n}^{p^k} + \beta \text{ for some } \beta \in [E, E].$$

Lemma 1.3.2 : Let K be a field of characteristic p>0 and let$\alpha \in K[G]$be nilpotent.If $1 \in Supp(\alpha)$ then there exists $x \in Supp(\alpha)$ such that $x \neq 1$ and the order of x is a power of p.

Theorem 1.3.3 : Let K be a field of characteristic p>0 and G have no elements of order p. Then K[G] has no nonzero nil ideals.

Proof : Let I be any nonzero ideal in K[G] then there exists $\beta \in I$ such that $\beta \neq 0$. If $y \in Supp(\beta)$ then $\alpha = y^{-1}\beta \in I \; and \; 1 \in Supp(\alpha)$. Since α is nilpotent therefore by above lemma, there exists $x \in Supp(\alpha), x \neq 1$ such that order of x is a power of p. Thus G has an element $x \neq 1$ whose order is a power of p. But then G has an element of order p, a contradiction. Therefore K[G] has no nonzero nil ideals.

REFERENCES:

1. Connel,I.G. On the group rings.
2. Herstein,I.N. Non Commutative rings.
3. Lam,T.Y. Non Commutative rings.
4. Passman,D.S. Infinite Group rings.
5. Passman,D.S. The Algebraic Structure of Group Rings.