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On separation properties of IFS GENERATING SUB AND SUPER SELF-SIMILAR SETS

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Abstract

A compact set is self-similar set if it can be expressed as a union of similar copies of itself. An iterated function system is a complete metric space with a finite set of contraction mappings which is a mathematical tool to construct fractal shapes. There are different forms of self-similarity in nature and in mathematics. In this paper we discuss the nature of contraction mappings which is used to construct sub and super self-similar sets. The separation properties of the iterated function systems which generates these sets are also studied.

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***Keywords**:* Self similar sets, Sub self similar sets, Super self-similar sets, Iterated function systems

1. INTRODUCTION

Self-similarity in fractals is currently an area of intense research. A wide range of fractals are self-similar in the sense that they are made up of parts which are similar, in some way, to the whole. Self-similarity is not only a property but it actually defines a fractal. Many natural objects exhibit self-similar properties to some extent. The different forms of self-similarity include sub, super, weak, quasi and partial self-similarity. A general account of fractals is given by Mandelbrot [2], whilst a more mathematical approach may be found in Falconer [8, 10, 11]. Self-similarity was mathematically presented by Hutchinson in 1981[7] and an enormous literature has developed on representation of self-similar sets, its dimension and measures [4, 5, 8]. The concepts of sub self-similarity and super self-similarity were introduced by Falconer in 1995[9] and later studied by [16, 17]. An effective method for producing fractal shapes is with iterated function systems by using deterministic or chaos game algorithms. This approach has been used to produce natural fractal shapes, find fractal interpolation functions to a given data, to obtain fractal approximations to functions and even to visualize discrete arbitrary sequences [1, 3, 6, 13-15].

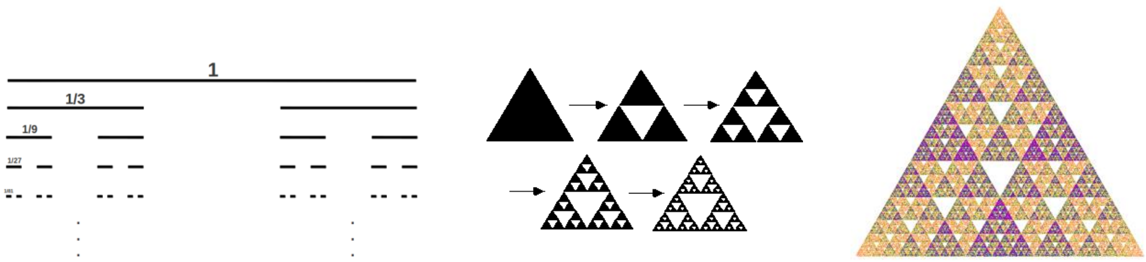
2. PRELIMINARIES

2.1 Basic definitions

Some of the basic definitions and results [8, 12] required for the development of the paper is discussed in this section.

We work in a fixed Euclidean space R^n. Let $H(R^n)$ denote the set of non-empty, compact subsets of R^n. We define $h(A,B)=\max\{\sup_{x\in A}\{dist(x,B)\},\sup_{y\in B}\{dist(y,A)\}$ which is the Hausdorff metric on R^n where $A,B\in H(R^n)$. We can see that $H(R^n)$ is complete in the metric h. A mapping $S: R^n \to R^n$ is called a contraction on R^n if there is a number c with $0 < c < 1$ such that $\|S(x) - S(y)\| \le c\|x - y\|, \forall x, y \in R^n$. Clearly, any contraction is a continuous mapping. If equality holds, ie; if $\|S(x) - S(y)\| = c\|x - y\|$, then S transforms sets into geometrically similar ones and we call S a similarity. A (hyperbolic) IFS consists of a complete metric space (X, d) together with a finite set of contraction mappings $w_j : X \to X$ with respect to the contraction factors s_j , for $j = 1.2,\ldots N$ The notation of the IFS just defined is $\{X; w_j, j = 1,2,\ldots N\}$ and its contraction factor is $s = \max\{s_j : j = 1,2,\ldots N\}$. Let $\{X; w_j, j = 1,2,\ldots N\}$ be a hyperbolic IFS with contraction factor *s*. Then the transformation $W: H(X) \to H(X)$ defined by $W(B) = \bigcup_{j=1}^{N} w_j(B)$ for all $B \in H(X)$, is a contraction mapping on the complete metric space $(H(X), h)$ with contraction factor *s*. That is, $h(W(B), W(C)) \le s.h(B, C)$ for all $B, C \in H(X)$. Its unique fixed point, $A \in H(X)$ obeys $A = W(A) = \bigcup_{j=1}^{N} w_j(A)$ and is given by $A = \lim_{n \to \infty} W^{on}(B)$ for any $B \in H(X)$. The fixed point $A \in H(X)$ described here is called the attractor of the IFS. There are many classes of IFS of special interest. If the contraction mappings $\{w_j, j = 1,2,\ldots N\}$ are similarities, the attractor E is called self-similar, if they are affine transformations E is called self-affine, and if they are conformal transformations then E is called self-conformal.

An IFS with some attractor A is said to be overlapping if for some w_i, w_j, there exist an open set O such that $O \subset w_i(A) \cup w_j(A)$. It is totally disconnected if $w_i(A) \cap w_j(A) = \phi$ for $i \ne j$. It is a just touching IFS if it is not totally disconnected yet its attractor contains an open set O such that $w_i(A) \cap w_j(A) = \phi$ for $i \ne j$ and $\bigcup_{i=1}^{N} w_i(O) \subset O$. These two conditions together are known as open set conditions.



(a) (b) (c)

Figure 1. Attractors of (a) totally disconnected (b) just touching and (c) overlapping IFS

2.2 Self similar sets

Given $m \geq 2$, and contracting similarities $S_i : R^n \to R^n$, (i =1, 2, 3, . . . , m) there exists a unique non-empty compact set $E \subseteq R^n$ satisfying

$$E = \bigcup_{i=1}^{m} S_i(E) \qquad (1)$$

This set E is called self-similar, or self-similar for $\{S_1,\ldots,S_m\}$ if the similarity transformations need to be emphasized. Thus a self-similar set is a metric space that is the union of scaled versions of itself, with scaling factor less than one. In his paper on sub self-similar sets, Falconer [9] gives a generalization of self-similar sets by relaxing the inequality in eq. (1) to inclusion. ie. With $m \geq 2$, and contracting similarities $S_i : R^n \to R^n$, (i = 1, 2, 3, . . . , m), a non-empty compact set $E \subseteq R^n$ is called sub self-similar for $\{S_1,\ldots,S_m\}$, if

$$E \subseteq \bigcup_{i=1}^{m} S_i(E) \qquad (2)$$

Later, Falconer [9] also defined super self similar set by reversing the inclusion. A non-empty compact set $E \subseteq R^n$ is called super self similar, if there are contracting similarities $S_i : R^n \to R^n$, (i = 1, 2, 3, . . . , m) for $m \geq 2$ such that

$$E \supseteq \bigcup_{i=1}^{m} S_i(E) \qquad (3)$$

3. MAIN RESULTS

Definition 3.1: A non-empty compact set $E \subseteq R^n$ is called **strictly sub self-similar**, if there are contracting similarities $S_i : R^n \to R^n$, (i = 1, 2, 3, . . . , m) for $m \geq 2$ such that

$$E \subset \bigcup_{i=1}^{m} S_i(E) \qquad (4)$$

and is **strictly super self similar** if we reverse the set inclusion in eq. (4).

It is clear from the definition of contracting similarity transformation that it is a composition of isometry and homothety. A transformation which preserves distances is called an isometry. A transformation $\mu_c : R^n \to R^n$ is a homothety if $\mu_c(x) = cx, (c \geq 0)$. A transformation $\tau_b : R^n \to R^n$ is a translation if $\tau_b(x) = x - b$. Translation is an isometry.

Proposition 3.2:[8] $S : R^n \to R^n$ is a similarity transformation iff $S = \mu_c o \tau_b o O$ for some homothety μ_c, translation τ_b, and orthonormal transformation O.

And in a contraction, the scaling factor c is such that $0 < c < 1$. Thus a contracting similarity can transform points of a given set to points inside or outside that set.

Definition 3.3: Let (X, d) is a complete metric space and $E \in H(X)$. If a contracting similarity S maps points of E to the points of E itself, then we call it a contracting sub similarity transformation for the set E. In this case, $S(E) \subseteq E$.

Definition 3.4: Let (X, d) is a complete metric space and $E \in H(X)$. If a contracting similarity S maps points of E to points which also lies outside E, then we call it a contracting super similarity transformation for the set E.

Thus contracting sub similarities maps a set into itself whereas contracting super similarities maps a set to its outside. For example, consider the set $E = [0,1]$ then the transformation $S_1 = \frac{1}{2}x$ maps E to $[0, \frac{1}{2}]$ and hence is a contracting sub similarity, whereas the transformation $S_2 = \frac{1}{2}x + 1$ maps E to $[0, \frac{3}{2}]$ which contains points outside E and hence is a contracting super similarity for E. Since the contracting super similarity transformation maps points of a set which also lies in its complement we can have two types of such transformation. One which maps a set to some points of the set and its complement and the other which maps it entirely to its complement. Accordingly, we make the following definition.

Definition 3.5: Let (X, d) is a complete metric space and $E \in H(X)$. If a contracting super similarity transformation S maps points of E to points which also lies outside E, then we call it a **contracting partial super similarity transformation** for the set E. Here $S(E)$ contains some points of E and some points of E^C where E^C is the complement of E.

Definition 3.6:. Let (X,d) is a complete metric space and $E \in H(X)$. If a contracting super similarity transformation S maps points of E to its complement, then we call it a **contracting exact super similarity transformation** for the set E. In this case $S(E)$ does not contain any point of E. i.e. $S(E) \subseteq E^C$.

For example consider the set $F = [1,2]$ and the contracting similarities $S_1 = \frac{x+2}{4}$, $S_2 = \frac{x}{3}$. Here S_1 is a contracting partial super similarity transformation and S_2 is a contracting exact super similarity transformation for the set F.

Theorem 3.7: Let (X,d) is a complete metric space and $E \in H(X)$ be a sub self-similar set which is the attractor of an IFS $\{X; S_j, j = 1,2,....N\}$, then the IFS is either just touching or overlapping. Moreover, if E is strictly sub self-similar then at least one of the contraction mapping S_j is a contracting partial super similarity transformation for the set E.

Proof: Since E is sub self-similar, by definition $E \subseteq \bigcup_{i=1}^{N} S_i(E)$ (1)

So $x \in E \Rightarrow x \in \bigcup_{1}^{N} S_i(E) \Rightarrow x \in S_j(E)$ for some j. which implies each point of E is contained in some $S_j(E)$. This implies $S_i(E) \cap S_j(E) \neq \phi$ for $i \neq j$. Thus the IFS is not totally disconnected. ie. The IFS is either just touching or overlapping.

Now, if E is strictly sub similar, the inclusion in eq.(1) becomes strict which implies the set E is fully contained in the union of the sets on the right hand side of the inclusion. For this, the union $\bigcup_{i=1}^{N} S_i(E)$ must contain some points in E^C which implies at least one of the contraction mapping S_j is contracting partial similarity transformation for the set E. This completes the proof.

Theorem 3.8: Let (X,d) is a complete metric space and $E \in H(X)$ be the attractor of an IFS $\{X; S_j, j = 1,2,....N\}$. If the IFS is either just touching or overlapping and at least one of the contraction mapping S_j is a contracting partial super similarity transformation for the set E, then E is a strictly sub self similar set.

Proof: Since the IFS is just touching or overlapping, it is not totally disconnected. ie. $S_i(E) \cap S_j(E) \neq \phi$ for $i \neq j$. This implies every element in E is contained in some $S_j(E)$ for $j = 1,2,....N$. Thus we get $E \subseteq \bigcup_{i=1}^{N} S_i(E)$. Also, one of the contraction mapping, say S_j is a contracting partial similarity transformation for the set E. This means $S_j(E)$ contains some points of E^C $E \subset \bigcup_{i=1}^{N} S_i(E)$ which implies E is a strictly super self-similar set.

Theorem 3.9: Let (X,d) is a complete metric space then the attractor $E \in H(X)$ of an IFS $\{X; S_j, j = 1,2,....N\}$ is a super self-similar set iff each S_j will be a contracting sub similarity transformation for the set E.

Proof: The proof follows from the definition of a super self-similar set and contracting sub similarity transformation which can be seen in [17].

The IFS which generates a super self-similar set can be totally disconnected, just touching or overlapping type. Accordingly, the attractor will be totally disconnected, just touching or overlapping set respectively.

CONCLUSION

Different forms of self-similarity like sub, super, strictly sub and strictly super self-similarity are discussed in the paper. The nature of contraction similarities which generates sub and super self-similar sets are discussed and the separation properties of the generating IFS are studied.

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