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# Poisson Summation Formula for Two dimensional Fractional Fourier Transform

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**Abstract:** The Fractional Fourier Transform belongs to the class of time–frequency representations that have been extensively used by the signal processing community. Sampling is an operation which only affects the time-axis of a signal. . Poisson summation formula is useful in electromagnetic problems engaging cylindrical geometries, analysis of wave radiation and scattering from large finite arrays, fast linear convolution, modified-fibonacci antenna arrays, a new technique for calculating Fourier integrals.

In this paper we have discussed on sampling theorem and Poisson summation formula for two dimensional fractional Fourier Transform. It has many applications in engineering, physics and number theory.

**Keywords-** Testing function space, Fractional Fourier transform, Two dimensional fractional Fourier transform, Sampling Theorem, Poisson summation formula.

## I. Introduction

Some fractional transforms arise under consideration of different problem : description of paraxial diffraction in free space and in a quadrant refractive index medium, resolution of nonstationary Schrodinger equation in quantum mechanics, phase retrieval and so forth. Other fractional transform can be constructed for their own sake, even if their direct application may not be obvious yet [2].

The fractional Fourier transform is generalization of classical Fourier transform. The canonical fractional Fourier transform was introduced more than 60 years ago in the mathematical literature [3], after that, it was reinvented for applications in quantum mechanics [4] [8], optics [5] [6] [7], and signal processing [6]. In 1929, the fractional power of FT operator appeared in the mathematical literature. The Fractional Fourier Transform (FRFTs) are commonly called as rotational Fourier transform or angular Fourier transform in some research papers. The applications of FRFT are quantum mechanics , signal processing , pattern recognition, and optical, video and audio processing . In optics, the continuous FRFT is implemented . Thus in short, it is proved fact that difference provides a richer solution set as compared to their continuous limit differential equations [11]. Fractional Fourier transform has a good application value in the seismic noise removal [9]. The use of fractional Fourier transform offers additional degrees of freedom to enlarge the key size, thus enhancing the level of security so it is used in optical image watermarking scheme [12]. The Poisson summation formula describes the fundamental duality between periodization and decimation operators under the Fourier transform [13]. Poisson summation is generally associated with the physics of periodic media, such as heat conduction on a circle. It has been successfully applied to the problem of the radiation of a semiinfinite array of periodic line sources that obey Floquet's theorem, i.e., equal amplitude currents with a progressive phase factor [14].

In this paper we developed sampling theorem for two dimensional fractional Fourier transform. Sampling theorem determines the necessary conditions which allow us to change an analog signal to a discrete one, or vice versa, without loss of information.

## II. Definitions-

### 2.1 Two Dimensional Fractional Fourier Transform

The two-dimensional fractional Fourier transform with parameters $\alpha$ of $f(x,y)$ denoted by FRFT{ $f(x,y)$} performs a linear operation, given by the integral transform.

FRFT { $f(x,y)\} = F_\alpha(\xi,\eta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)K_{\alpha,\theta}(x,y,\xi,\eta)dxdy$ ----(2.1.1)

where $K_\alpha(x,y,\xi,\eta) = \sqrt{\frac{1-icot\alpha}{2\pi}}\, e^{\frac{1}{2sin\alpha}[(x^2+y^2+\xi^2+\eta^2)cos\alpha-2(x\xi+y\eta)}$

$= C_{1\alpha}e^{iC_{2\alpha}[(x^2+y^2+\xi^2+\eta^2)cos\alpha-2(x\xi+y\eta)}$

where $C_{1\alpha} = \sqrt{\frac{1-icot\alpha}{2\pi}}$, $C_{2\alpha} = \frac{1}{2sin\alpha}$ $0 < \alpha < \frac{\pi}{2}$ ---(2.1.2)

### 2.2 Testing function space-

An infinitely differentiable complex valued smooth function $\emptyset(x,y)$ on $R^n$ belongs to $E(R^n)$, if for each compact set $I \subset S_{a,b}$, $J \subset S_{c,d}$ where

$S_{a,b} = \{x,y: x,y\epsilon R^n, |x| \le a, |y| \le b, a > 0, b > 0\}$

$\gamma_{E,m,n}[\emptyset(x,y)] = \sup_{x,y\epsilon I}|D_{x,y}^{m,n}\emptyset(x,y)| < \infty$ ---(2.2.1)

Thus $E(R^n)$ will denote the space of all $\emptyset(x,y)\epsilon E(R^n)$ with compact support contained in $S_{a,,b}$.

Note that the space $E$ is complete and therefore a Frechet space. Moreover, we say that $f(x,y)$ is a two dimensional fractional Fourier transformable if it is a member of $E$ .

### 2.3 Distributional Two Dimensional Fractional Fourier Transform (FRFT)

The two dimensional distributional Fractional Fourier transform of $f(x,y)\epsilon E^*(R^n)$ can be defined by

$FRFT\{f(x,y)\} = F_\alpha(\xi,\eta) = \langle f(x,y), K_\alpha(x,y,\xi,\eta)\rangle$ ---(2.3.1)

where, $K_\alpha(x,y,\xi,\eta) = \sqrt{\frac{1-icot\alpha}{2\pi}}\, e^{\frac{1}{2sin\alpha}[(x^2+y^2+\xi^2+\eta^2)cos\alpha-2(x\xi+y\eta)} = C_{1\alpha}e^{iC_{2\alpha}[(x^2+y^2+\xi^2+\eta^2)cos\alpha-2(x\xi+y\eta)}$

where $C_{1\alpha} = \sqrt{\frac{1-icot\alpha}{2\pi}}$, $C_{2\alpha} = \frac{1}{2sin\alpha}$, $0 < \alpha < \frac{\pi}{2}$, . ---(2.3.2)

Right hand side of equation (2.3.1) has a meaning as the application of $f(x,y)\epsilon E^*(R^n)$to $K_\alpha(x,y,\xi,\eta)\epsilon E$. It can be extended to the complex space as an entire function given by

$FRFT\{f(x,y)\} = F_\alpha(\xi',\eta') = \langle f(x,y), K_\alpha(x,y,\xi',\eta')\rangle$ ---(2.3.3)

The right hand side is meaningful because for each $\xi',\eta'\epsilon C^n$, $K_{\alpha,\theta}(x,y,\xi',\eta')\epsilon E$ as a function of $x,y$.

## III. Poisson Summation Formulae Associated with Fourier Transform:

Using the concept of Poisson summation formula for the generalized Fourier transform as per [15, 16], here we obtained the Poisson summation formula for Fourier transform.

### 3.1 Poisson Summation Formula for Fourier Transform:

By the definition of two dimensional fractional Fourier transform

$$\mathcal{F}\{f(x,y)\}(p,q)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)e^{-i(wx+sy)}dxdy=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)e^{-iwx}e^{-isy}dxdy$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)e^{-i(\rho+i\sigma)x}e^{-i(\Omega+i\psi)y}dxdy$$

where, $w=\rho+i\sigma,\ s=\Omega+i\psi$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[f(x,y)e^{-i(\rho x+\Omega y)}]e^{\sigma x+\psi y}dxdy$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[f(x,y)e^{\sigma x+\psi y}]e^{-i(\rho x+\Omega y)}dxdy$$

$$=F\{g(x,y)\}(\rho,\Omega)=G(\rho,\Omega) \qquad (3.1.1)$$

We assume that $f(x,y)$ is the function such that Fourier transform has compact support say $\Omega_\alpha$ i.e. $|u|>\Omega_\alpha$ then $FRFTf(x,y)(p,q)=0$.
Now we apply Poisson's formula for Fourier Transform and write

$$\sum_{x=-\infty}^{\infty}\sum_{y=-\infty}^{\infty}g(x+k\tau,y+l\xi)=\frac{1}{\tau\xi}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}G(\frac{n}{\tau},\frac{m}{\xi})\,e^{i(\frac{nx}{\tau}+\frac{my}{\xi})} \qquad (3.1.2)$$

where G denotes the conventional Fourier Transform of $g(x,y)$

$$\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}f(x+k\tau,y+l\xi)\,e^{\sigma(x+k\tau)+\psi(y+l\xi)}=\frac{1}{\tau\xi}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\mathcal{F}\{f(x,y)\}(\frac{n}{\tau},\frac{m}{\xi})\quad e^{i(\frac{nx}{\tau}+\frac{my}{\xi})}$$

$$\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}f(x+k\tau,y+l\xi)\,e^{\sigma(k\tau)+\psi(l\xi)}=\frac{e^{-(\sigma x+\psi y)}}{\tau\xi}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\mathcal{F}\{f(x,y)\}(\frac{n}{\tau},\frac{m}{\xi})\,e^{i(\frac{nx}{\tau}+\frac{my}{\xi})} \qquad (3.1.3)$$

### 3.2 Poisson Summation Formulae for two dimensional Fractional Fourier Transform

Using the above Poisson Summation formulae for the two dimensional Fractional Fourier Transform we develop the Poisson Summation formula two dimensional Fractional Fourier Transform.

**Theorem-** Let, $f(x,y)$ be the function having compact support in two dimensional Fractional Fourier Transform domain i. e. If $[FRFT\{f(x,y)\}]=FRFT(p,q)$ and $|p|>\Omega$ and $|q|>\Omega$ , $FRFT(p)=0, FRFT(q)=0$ then

**Proof-** By the definition of FRFT we have,

$$FRFT(p,q)=\sqrt{\frac{1-icot\alpha}{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)\,e^{\frac{i}{2sin\alpha}[(x^2+y^2+p^2+q^2)cos\alpha-2(xp+yq)}dxdy$$

$$\sqrt{\frac{2\pi}{1-icot\alpha}}FRFT(p,q)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)\,e^{\frac{i}{2}(x^2+y^2)cot\alpha+\frac{i}{2}(p^2+q^2)cot\alpha-i(xp+yq)cosec\alpha}dxdy$$

$$\sqrt{\frac{2\pi}{1-icot\alpha}}e^{\frac{-i}{2}(p^2+q^2)cot\alpha}FRFT(p,q)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{f(x,y)\,e^{\frac{i}{2}(x^2+y^2)cot\alpha}\}e^{-i(xp+yq)cosec\alpha}dxdy$$

$$= \mathcal{F}\{g(x,y)\}(pcosec\alpha, qcosec\alpha) = G(V,T) \quad \text{---(3.2.1)}$$

$\because V = pcosec\alpha,\ T = qcosec\alpha,\ p = \frac{V}{cosec\alpha} = Vsin\alpha,\ q = \frac{T}{cosec\alpha} = Tsin\alpha$

where, $g(x,y) = f(x,y)\, e^{\frac{i}{2}(x^2+y^2)cot\alpha}$ --------(3.2.2)

$\therefore G(V,T) = \sqrt{\frac{2\pi}{1-icot\alpha}}\, e^{\frac{i}{2}(p^2+q^2)cot\alpha} FRFT(p,q) \quad = \sqrt{\frac{2\pi}{1-icot\alpha}}\, e^{\frac{i}{2}(V^2sin^2\alpha+T^2sin^2\alpha)cot\alpha} FRFT(Vsin\alpha, Tsin\alpha) =$

$\mathcal{F}\{g(x,y)\}(V,T)$ -------(3.2.3)

Poisson Formula for two dimensional Fourier transform (3.1.3)

$$\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} g(x+k\tau, y+l\xi)\, e^{\sigma(k\tau)+\psi(l\xi)} = \frac{e^{-(\sigma x+\psi y)}}{\tau\xi}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \mathcal{F}\{f(x,y)\}(\tfrac{n}{\tau},\tfrac{m}{\xi})\, e^{i(\frac{nx}{\tau}+\frac{my}{\xi})}$$

$$= \frac{e^{-(\sigma x+\psi y)}}{\tau\xi}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} G(\tfrac{n}{\tau},\tfrac{m}{\xi})\, e^{i(\frac{nx}{\tau}+\frac{my}{\xi})}$$

From (3.2.1) and (3.2.2)

$$\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} e^{\sigma(k\tau)+\psi(l\xi)} f(x+k\tau, y+l\xi)\, e^{\frac{i}{2}[(x+k\tau)^2+(y+l\xi)^2]cot\alpha}$$

$$= \sqrt{\frac{2\pi}{1-icot\alpha}}\frac{e^{-(\sigma x+\psi y)}}{\tau\xi}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} FRFT(\tfrac{n}{\tau}sin\alpha, \tfrac{m}{\xi}sin\alpha)\, e^{\frac{-i}{2}[(\frac{n}{\tau})^2+(\frac{m}{\xi})^2]sin^2\alpha cot\alpha} e^{i(\frac{nx}{\tau}+\frac{my}{\xi})}$$

$$\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} e^{\sigma(k\tau)+\psi(l\xi)} f(x+k\tau, y+l\xi)\, e^{\frac{i}{2}[(x^2+2xk\tau+k^2\tau^2)+(y^2+2yl\xi+l^2\xi^2)]cot\alpha}$$

$$= \sqrt{\frac{2\pi}{1-icot\alpha}}\frac{e^{-(\sigma x+\psi y)}}{\tau\xi}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} 2DFRFT(\tfrac{n}{\tau}sin\alpha, \tfrac{m}{\xi}sin\alpha)\, e^{-\frac{i}{2}[(\frac{n}{\tau})^2+(\frac{m}{\xi})^2]sin^2\alpha cot\alpha} e^{i(\frac{nx}{\tau}+\frac{my}{\xi})}$$

$$\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} e^{\sigma(k\tau)+\psi(l\xi)} f(x+k\tau, y+l\xi)\, e^{\frac{i}{2}[(2xk\tau+k^2\tau^2)+(2yl\xi+l^2\xi^2)]cot\alpha}$$

$$= \frac{e^{-(\sigma x+\psi y)}e^{-\frac{i}{2}(x^2+y^2)cot\alpha}}{\tau\xi} C_\alpha \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} FRFT(\tfrac{n}{\tau}sin\alpha, \tfrac{m}{\xi}sin\alpha)\, e^{-\frac{i}{2}[(\frac{n}{\tau})^2+(\frac{m}{\xi})^2]sin^2\alpha cot\alpha} e^{i(\frac{nx}{\tau}+\frac{my}{\xi})}$$

where, $C_\alpha = \sqrt{\frac{2\pi}{1-icot\alpha}}$---(3.2.4)

### 3.3 Poisson Summation Formulae for two dimensional Fractional Fourier Transform when $x = y = 0$

Putting $x = y = 0$ in equation (3.2.4). It becomes

$$\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} e^{\sigma(k\tau)+\psi(l\xi)} f(k\tau, l\xi)\, e^{\frac{i}{2}[(k^2\tau^2)+(l^2\xi^2)]cot\alpha}$$

$$= \frac{C_\alpha}{\tau\xi}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} e^{-\frac{i}{2}[(\frac{n}{\tau})^2+(\frac{m}{\xi})^2]sin^2\alpha cot\alpha} 2DFRFT(\tfrac{n}{\tau}sin\alpha, \tfrac{m}{\xi}sin\alpha) \quad \text{---(3.3.1)}$$

Equation (3.2.4) and (3.3.1) can be seen as Poisson Summation formula associated with fractional Fourier transform of order $\alpha$, it is clear from the above equation that the infinite sum of the periodic replica of function $f(x,y)$ is equal to infinite sum of its fractional Fourier transform.

### 3.4 Corollary 1

If $f(x,y)$ is in $\Omega$ having compact support in two dimensional fractional Fourier transform domain and $\frac{sin\alpha}{\tau} > \Omega, \frac{sin\alpha}{\xi} > \Omega$, then $\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} e^{\sigma(k\tau)+\psi(l\xi)} f(x+k\tau, y+l\xi)\, e^{\frac{i}{2}[(2xk\tau+k^2\tau^2)+(2yl\xi+l^2\xi^2)]cot\alpha}$

$$=\frac{C_\alpha e^{-[\sigma x+\psi y]}}{\tau\xi} e^{-\frac{i}{2}[(x)^2+(y)^2]cot\alpha} FRFT(0,0)$$

**Proof-**

$f(x, y)$ is in $\Omega$ having compact support in two dimensional fractional Fourier transform domain and $\frac{sin\alpha}{\tau} > \Omega, \frac{sin\alpha}{\xi} > \Omega$ then $FRFT\left(\frac{n}{\tau}sin\alpha, \frac{m}{\xi}sin\alpha\right) = 0$ when $n \neq 0, m \neq 0$

From equation (3.2.4)

$$\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} e^{\sigma(k\tau)+\psi(l\xi)} f(x+k\tau, y+l\xi)\, e^{\frac{i}{2}[(2xk\tau+k^2\tau^2)+(2yl\xi+l^2\xi^2)]cot\alpha}$$

$$= \frac{C_\alpha e^{-[\sigma x+\psi y]}}{\tau\xi} e^{-\frac{i}{2}[(x)^2+(y)^2]cot\alpha} \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} e^{-\frac{i}{2}[\left(\frac{n}{\tau}\right)^2+(\frac{m}{\xi})^2]sin^2\alpha cot\alpha}$$

$$FRFT(\frac{n}{\tau}sin\alpha, \frac{m}{\xi}sin\alpha)\, e^{i(\frac{nx}{\tau}+\frac{my}{\xi})}$$

$\frac{sin\alpha}{\tau} > \Omega, \quad \left|\frac{nsin\alpha}{\tau}\right| > |n\Omega| > \Omega, \quad ,\forall n$ from $-\infty$ to $\infty$

$\frac{sin\alpha}{\xi} > \Omega \quad \left|\frac{msin\alpha}{\xi}\right| > |m\Omega| > \Omega \quad ,\forall m$ from $-\infty$ to $\infty$

$n = 0, m = 0$

$$\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} e^{\sigma(k\tau)+\psi(l\xi)} f(x+k\tau, y+l\xi)\, e^{\frac{i}{2}[(2xk\tau+k^2\tau^2)+(2yl\xi+l^2\xi^2)]cot\alpha}$$

$$=\frac{C_\alpha e^{-[\sigma x+\psi y]}}{\tau\xi} e^{-\frac{i}{2}[(x)^2+(y)^2]cot\alpha} FRFT(0,0)$$

**Conclusion :**

In this paper we represented the Poisson summation formula of two dimensional fractional Fourier transform i.e. the generalized form of Poisson sum formula related with two dimensional fractional Fourier transform. Also sampling theorem of two dimensional fractional Fourier transform, we observed that two dimensional fractional Fourier transform of 0-periodic with $(\alpha - \frac{\pi}{2})$ compact support, can be calculated using ( 2k+1) samples of the function in time domain, where k is the order of positive highest nonzero harmonic component in conventional Fourier domain. These concepts of two dimensional Fractional Fourier transform have been found to be useful in digital control, speech recognition technology, electromagnetic problem etc.

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