International Journal of Research in Engineering and Applied Sciences(IJREAS)
Available online at http://euroasiapub.org/journals.php
Vol. 7 Issue 6 , June-2017, pp. 69~76
ISSN (O): 2249-3905, ISSN(P): 2349-6525 | Impact Factor: 7.196

Inversion Theorem of Distributional Fourier-Stieltjes Transform

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Abstract

Aim of this paper is to generalize the Fourier-Stieltjes transform in the distributional sense. The generalized function is one of the most important branch of mathematics that has enormous application in practical fields. Especially, its applications to the theory of distribution and signal processing are very much noteworthy. Fourier Transform has many applications in several areas including partial differential equations (PDE), quantum mechanics, Spectroscopy, signal processing, Radar system etc. and Stieltjes transform have enormous application in the field of random matrix theory, to solve the differential equations and many more. Together, Fourier-Stieltjes transform have applicable in the theory of Probability, for practical purpose they are more useful.

In this paper, we have presented the inversion and uniqueness theorem for the distributional Fourier-Stieltjes transform and some lemmas useful to prove inversion theorem.

Keywords: Fourier Transform, Stieltjes Transform, Fourier-Stieltjes Transform.

INTRODUCTION

The importance of integral transforms is that they provide powerful operational methods for solving initial value problems and initial-boundary value problems for linear differential and integral equations. Mathematical methods having greater demand to provide both theory and applications for science and engineering, the utility and interest of integral transforms seems more clearly established than ever, integral transforms have many mathematical and physical applications [2]. Integral transforms, viz. Fourier, Laplace, Hankel, Stieltjes, Mellin and the wavelet transform among other, have been used to obtain solutions of various types of differential and integral equations [3].

Mathematics is present in the natural occurrence of the ratios and intervals found in modern tuning systems. The Fourier Transform opens up practical possibilities to model and define sound using Fourier analysis [5]. The Fourier transform is a widely used tool for many applications. Its value in physics is best described by Lord Kelvin [1]. The development and extension to generalized functions of the Fourier transform became a remarkably powerful tool in the theory of partial differential equations. The Fourier transform is an important image processing tool which is used to decompose in image into its sine and cosine components. We can apply Fourier transform method for grayscale image with different resolutions [8].

The Stieltjes transform was first introduced by T.S. Stieltjes in connection to the moment problem for a semi-infinite interval. Since then it has been investigated and found to be useful in many different areas such as continued fractions, probability and signal processing etc [4]. Pandey also discussed the two-point Stieltjes transform with application; it is found that stieltjes transform is more suitable for the study [7].

In the present paper we have extended the Fourier-Stieltjes transform and in [9, 10] the conventional Fourier-Stieltjes transform of a complex valued smooth function $f(t,x)$ is defined by the convergent integral.

$$FS(s,p) = FS\{f(t,x)\} = \int_0^\infty \int_0^\infty f(t,x)\; e^{-ist}\; (x+y)^{-p}\; dt\; dx$$

Where, t and x are positive real numbers.

In the present work, we have presented the inversion and uniqueness theorem for the distributional Fourier-Stieltjes transform and some lemmas related to inversion theorem.

1. Some required Lemma's to prove inversion theorem

1.1 Lemma

Let $FS\{f(t.x)\} = F(s,y)$, $s>0$ and $\rho_1 < Re\; y < \rho_2$. For $t,x \in \Omega$ and $\phi(t,x) \in \mathcal{D}(\Omega)$ set,

$$\psi(s,y) = \int_{-\infty}^\infty \int_0^\infty \phi(t,x)\, e^{-ist} (x+y)^{-p}\; dt\; dx \qquad (1.1.1)$$

Then for any pair of real numbers r and τ with $-\infty < r < \infty$ and $0 < \tau < \infty$

$$\int_{-r}^{r}\int_0^\tau \langle f(u,v), \tfrac{pe^{-ist}}{(v+y)^{p+1}}\rangle\, \psi(s,y)\; ds\; dw = \langle f(u,v), \int_{-r}^{r}\int_0^\tau \psi(s,y) \tfrac{pe^{-ist}}{(v+y)^{p+1}} ds\; dw\rangle \qquad (1.1.2)$$

Where $y = \rho + iw$, also s and ρ are fixed real numbers such that $s>0, \rho_1 < \rho < \rho_2$ respectively.

Proof: For $\phi(t,x) = 0$ on Ω, the proof is trivial. Let $\phi(t,x) \neq 0$ on Ω, since $F(s,y)$ is analytic on Ω and $\psi(s,y)$ is entire, the integral on right hand side of equation (1.1.2) exists. First we shall prove that,

$$G(u,v) = \int_{-r}^{r}\int_0^\tau \psi(s,y)\; \frac{pe^{-isu}}{(v+y)^{p+1}}\; ds\; dw, \qquad (1.1.3)$$

as a function of (u,v) belongs to FS_α. For consider,

$$\gamma_{k,l,q,h} G(u,v) = \sup_{\substack{0<u<\infty \\ 0<v<\infty}} \left| u^k (1+v)^l D_u^q (vD_v)^h G(u,v) \right|$$

$$= \sup_I \left| u^k (1+v)^l D_u^q (vD_v)^h \int_{-r}^{r}\int_0^\tau \psi(s,y)\; \frac{pe^{-isu}}{(v+y)^{p+1}}\; ds\; dw \right|.$$

By the smoothness of the integral we may carry the operator $D_u^q (vD_v)^h$ under the integral sign. Hence we get,

$$\gamma_{k,l,q,h} G(u,v) = \sup_I \left| u^k (1+v)^l \int_{-r}^{r}\int_0^\tau D_u^q (vD_v)^h \psi(s,y)\; \frac{pe^{-isu}}{(v+y)^{p+1}}\; ds\; dw \right|$$

$$= \sup_I \left| \int_{-r}^{r}\int_0^\tau \sum_{j=1}^h \psi(s,y)\, u^k (1+v)^l (-is)^q e^{-isu} v^j (-1)^j P(p)\; \frac{1}{(v+y)^{p+j+1}}\; ds\; dw \right|$$

$$= \sup_I \left| \int_{-r}^{r}\int_0^\tau \sum_{j=1}^h \psi(s,y)\, u^k (1+v)^l (-s)^q e^{-isu} v^j (-1)^j P(p)\; \frac{1}{(v+y)^{p+j+1}}\; ds\; dw \right|$$

Where $P(p)$ is polynomial in p. The series on the right hand side is series of positive finite terms which is bounded by M_1 (say) for $u>0$ and $v>0$. Therefore $\gamma_{k,l,q,h} G(u,v) < \infty$, hence $G(u,v)$

belongs to $FS_\alpha(\Omega)$. Therefore the right hand side of equation (1.1.2) is meaningful.
Partitioning the path of integration on the straight line from $s=-r$ to $s=r$ into m interval each of length $\frac{2r}{m}$ and $y=\rho-i\tau$ to $y=\rho+i\tau$ into $2n$ intervals each of length $\frac{\tau}{n}$. Let $s_i=\sigma$ be any point in the i[th] interval and $y_j=\rho+iw$ be any point in the j[th] interval.
Consider

$$\phi_{m,n}(u,v)=\sum_{i=1}^{m}\sum_{j=1}^{n}\psi(s_i,y_j)\frac{pe^{-is_iu}}{(v+y_j)^{p+1}}\ \frac{2r}{m}\frac{\tau}{n}\ .$$

By applying $f(u,v)$ to above equation term by term, we get

$$\langle f(u,v),\phi_{m,n}(u,v)\rangle=\langle f(u,v),\sum_{i=1}^{m}\sum_{j=1}^{n}\psi(s_i,y_j)\frac{pe^{-is_iu}}{(v+y_j)^{p+1}}\ \frac{2r}{m}\frac{\tau}{n}\rangle$$

$$=\sum_{i=1}^{m}\sum_{j=1}^{n}\langle f(u,v),\frac{pe^{-is_iu}}{(v+y_j)^{p+1}}\rangle\ \ \psi(s_i,y_j)\frac{2r}{m}\frac{\tau}{n}$$

Since $\langle f(u,v),\frac{pe^{-is_iu}}{(v+y_j)^{p+1}}\rangle\,\psi(s,y)$ is a continuous function on $-r\le s\le r, 0<w\le\tau$ the sum on the right hand side tends to,

$$\int_{-r}^{r}\int_{0}^{\tau}\langle f(u,v),\frac{pe^{-isu}}{(v+y)^{p+1}}\rangle\ \ \psi(s,y)ds\ \ dw,\text{ as } m\to\infty,n\to\infty$$

Next, choose a,a',b and b' such that $\sigma_1<a<\sigma<b<\sigma_2$ and $\rho_1<a'<\rho<b'<\rho_2$. Since $f(t,x)\in FS_\alpha^*$ our lemma will be proved when $\phi_{m,n}(u,v)$ converges in FS_α to

$$\int_{-r}^{r}\int_{0}^{\tau}\psi(s,y)\ \ \frac{pe^{-isu}}{(v+y)^{p+1}}ds\ \ dw,$$

as$m\to\infty,n\to\infty$. So we need merely show that, for each fixed q and$h$, $|\phi_{m,n}(u,v)-G(u,v)|$ converges uniformly to zero on Ω as$m\to\infty,n\to\infty$.
Consider,

$$\gamma_{k,l,q,h}[\phi_{m,n}(u,v)-G(u,v)]=\sup_I\left|u^k(1+v)^lD_u^q(vD_v)^h[\phi_{m,n}(u,v)-G(u,v)]\right|$$

$$=\sup_I\left|u^k(1+v)^lD_u^q(vD_v)^h\left[\sum_{i=1}^{m}\sum_{j=1}^{n}\psi(s_i,y_j)\frac{pe^{-is_iu}}{(v+y_j)^{p+1}}\ \frac{2r}{m}\frac{\tau}{n}-\right.\right.$$
$$\left.\left.\int_{-r}^{r}\int_{0}^{\tau}\psi(s,y)\ \ \frac{pe^{-isu}}{(v+y)^{p+1}}ds\ \ dw\right]\right|$$

$$=\sup_I\left|(-i)^qu^k(1+v)^l\sum_{g=1}^{h}\sum_{i=1}^{m}\sum_{j=1}^{n}\psi(s_i,y_j)(s_i)^qa_gP(p)v^g\frac{e^{-is_iu}}{(v+y_j)^{p+g+1}}\ \frac{2r}{m}\frac{\tau}{n}-\right.$$
$$\left.(-i)^qu^k(1+v)^l\int_{-r}^{r}\int_{0}^{\tau}\sum_{g=1}^{h}\psi(s,y)\,a_g(s)^q\ \ P(p)v^g\frac{e^{-isu}}{(v+y)^{p+g+1}}ds\ \ dw\right|$$

(1.1.4)

Now, $\left|\sum_{g=1}^{h}u^k(1+v)^l\ \ v^g\frac{e^{-isu}}{(v+y)^{p+g+1}}\right|\to0,$ as $|u|\to\infty$ and $|v|\to\infty$ because $s>0$ and $Re\ \ p>l$. So for given $\epsilon>0$ we can select $T>0$ and $X>0$ as large that for all $|u|>T$ and $|v|>X$,

$$\left|\sum_{g=1}^{h}u^k(1+v)^l\ \ v^g\frac{e^{-isu}}{(v+y)^{p+g+1}}\right|<\frac{\epsilon}{3}\left[\int_{-r}^{r}\int_{0}^{\tau}\left|\sum_{g=1}^{h}\psi(s,y)\,a_g(s)^qP(p)\right|\ \ ds\ \ dw\right]^{-1}$$

since $\phi(t,x)\neq0$, the right hand side is finite. Now for all $|u|>T$ and $|v|>X$, the magnitude of the second term on the right side of equation (1.1.4) is bounded by $\frac{\epsilon}{3}$. Moreover again for $|u|>T$

and $|v| > X$, the magnitude of the first term on the right side of equation (1.1.4) is bounded by,

$\frac{\epsilon}{3}\left[\int_{-r}^{r}\int_{0}^{\tau}\left|\sum_{g=1}^{h}\psi(s,y)\,a_g(s)^q P(p)\right|\;ds\;dw\right]^{-1}\sum_{g=1}^{h}\sum_{i=1}^{m}\sum_{j=1}^{n}\left|\psi(s_i,y_j)(s_i)^q a_g P(p)\right|\;\frac{2r}{m}\frac{\tau}{n}$.

We can choose m_0, n_0 so large that for all $m > m_0$ and $n > n_0$, the last expression is less than $\frac{2\epsilon}{3}$. Therefore for all $|u| > T$ and $|v| > X$ and all $m > m_0$ and $n > n_0$,

$\gamma_{k,l,q,h}\left[\phi_{m,n}(u,v) - G(u,v)\right] < \epsilon$.

Finally $u^k(1+v)^l\sum_{g=1}^{h}\psi(s,y)\,a_g(s)^q P(p) v^g\,\frac{e^{-isu}}{(v+y)^{p+g+1}}$ is a uniformly continuous function of (t,x,s,w) on the domain $-T \le u \le T, 0 \le v \le X, -r \le s \le r$ and $0 \le w \le \tau$. Then in view of equation (1.1.4) there exist m_1 and n_1 such that for all $m > m_1$ and $n > n_1$, $\gamma_{k,l,q,h}\left[\phi_{m,n}(u,v) - G(u,v)\right] < \epsilon$ on $-T \le u \le T$ and $0 \le v \le X$ as well. Thus when $m > \max(m_0, m_1)$ and $n > \max(n_0, n_1)$, $\gamma_{k,l,q,h}\left[\phi_{m,n}(u,v) - G(u,v)\right] < \epsilon$ on $-\infty < u < \infty$ and $0 < v < \infty$ which gives the result.

1.2 Lemma

For $\phi \in \mathcal{D}(I)$, set $\psi(s,y)$ as in lemma 1.1 then,

$$\frac{1}{4\pi^2}\int_{-r}^{r}\int_{0}^{\tau}\psi(s,y)\;\frac{pe^{-isu}}{(v+y)^{p+1}}ds\;dw = \frac{-1}{2\pi^2}\int_{-\infty}^{\infty}\int_{0}^{\infty}\phi(t+u,x)\;\frac{\sin rt}{t}\frac{1}{(x-v)}dx\;dt.$$

Proof: We shall prove the result by justifying the steps in the following manipulations and by considering compact support of $\phi(t,x) \in \mathcal{D}(I)$.

$$\frac{1}{4\pi^2}\int_{-r}^{r}\int_{0}^{\tau}\psi(s,y)\;\frac{pe^{-isu}}{(v+y)^{p+1}}ds\;dw$$

$$= \frac{1}{4\pi^2}\int_{-r}^{r}\int_{0}^{\tau}\frac{pe^{-isu}}{(v+y)^{p+1}}\left[\int_{-\infty}^{\infty}\int_{0}^{\infty}\phi(t,x)e^{ist}\,(x+y)^{p-1}\;dt\;dx\right]\;ds\;dw$$

$$= \frac{1}{4\pi^2}\int_{-r}^{r}\int_{-\infty}^{\infty}\int_{0}^{\infty}\phi(t,x)\,pe^{-isu}e^{ist}\left[\int_{0}^{\tau}\frac{(x+y)^{p-1}}{(v+y)^{p+1}}dy\right]\;dx\,dt\;ds$$

By using the proper substitution and result from Convey [13 pp. 18] as $v \to \infty$ the bracket from right hand side tends to $\frac{1}{p(v-x)}$. Hence,

$$\frac{1}{4\pi^2}\int_{-r}^{r}\int_{0}^{\tau}\psi(s,y)\;\frac{pe^{-isu}}{(v+y)^{p+1}}ds\;dw$$

$$= \frac{1}{4\pi^2}\int_{-r}^{r}\int_{-\infty}^{\infty}\int_{0}^{\infty}\phi(t,x)\frac{pe^{is(t-u)}}{p(v-x)}\;dx\,dt\;ds$$

$$= \frac{1}{4\pi^2}\int_{-r}^{r}\int_{-\infty}^{\infty}\int_{0}^{\infty}\phi(t,x)\frac{e^{is(t-u)}}{(v-x)}\;dx\,dt\;ds$$

By using Zemanian [68, pp 69] above equation reduces to

$$= \frac{-1}{2\pi^2}\int_{-\infty}^{\infty}\int_{0}^{\infty}\phi(t+u,x)\;\frac{\sin rt}{t}\frac{1}{(x-v)}dx\;dt$$

1.3 Lemma

Let r be real number with $a < s < b$. Also let $\phi(t,x) \in \mathcal{D}(I)$, then $\pi^{-1}\int_{-\infty}^{\infty}\phi(t+u,x)\frac{\sin rt}{t}\;dt$ converges in FS_α to $\phi(u,x)$, as $r \to \infty$.

Proof: In the following assume that $r > 0$. It is known that,

$$\int_{-\infty}^{\infty} \frac{\sin rt}{t}\, dt = \pi \tag{1.3.1}$$

Thus our objective is to prove that for each $k = 0,1,2,3 \dots \dots \dots B_r(u,x)$ converges uniformly to zero on $-\infty < t < \infty$ and $r \to \infty$, where

$$B_r(u,x) = \frac{1}{\pi}\ u^k(1+x)^l D_u^q (xD_x)^h \int_{-\infty}^{\infty}[\phi(t+u,x) - \phi(t,x)]\ \frac{\sin rt}{t}\ dt$$

$$= \frac{1}{\pi}\{u^k(1+x)^l D_u^q (xD_x)^h \int_{-\infty}^{\infty}\phi(t+u,x)\ \frac{\sin rt}{t}\ dt -$$

$$u^k(1+x)^l D_u^q (xD_x)^h \int_{-\infty}^{\infty}\phi(t,x)\ \frac{\sin rt}{t}\ dt\}$$

$$= \frac{1}{\pi}\ \{\int_{-\infty}^{\infty} H(t,u,x)\ \sin rt\ \ dt\},$$

where,

$$H(t,u,x) = \frac{u^k(1+x)^l}{t}[D_u^q (xD_x)^h\phi(t+u,x) - D_u^q (xD_x)^h\phi(t,x)]\ . \tag{1.3.2}$$

Therefore

$$B_r(u,x) = \frac{1}{\pi}\ \{\int_{-\infty}^{-\delta} H(t,u,x)\ \sin rt\ \ dt + \int_{-\delta}^{\delta} H(t,u,x)\ \sin rt\ \ dt$$

$$+ \int_{\delta}^{\infty} H(t,u,x)\ \sin rt\ \ dt\}$$

$$B_r(u,x) = \frac{1}{\pi}\ \int_{-\infty}^{-\delta} H(t,u,x)\ \sin rt\ \ dt + \frac{1}{\pi}\int_{-\delta}^{\delta} H(t,u,x)\ \sin rt\ \ dt\ +$$

$$\frac{1}{\pi}\int_{\delta}^{\infty} H(t,u,x)\ \sin rt\ \ dt$$

$$= I_{1,r}(u,x) + I_{2,r}(u,x) + I_{3,r}(u,x)$$

where $I_{1,r}(u,x), I_{2,r}(u,x)$ and $I_{3,r}(u,x)$ denotes the quantities obtained by integrating over $-\infty < t < -\delta, -\delta < t < \delta$ and $\delta < t < \infty$ respectively where $\delta > 0$.

First consider $I_{2,r}(u,x)$: The function $H(t,u,x)$ is a continuous function of (t,u,x) for all u and $t \neq 0$. Moreover since ϕ is smooth, equation (1.3.2) tends to

$$u^k(1+x)^l D_t[\tfrac{1}{t} D_u^q (xD_x)^h\phi(t+u,x)]_{t=0} \tag{1.3.3}$$

As $t \to 0$. Upon assigning the value from above equation (1.3.3) to $H(0,u,x)$ we obtain a function $H(t,u,x)$ which is continuous everywhere. Since, ϕ is of bounded support, $H(t,u,x)$ is bounded on the domain $\{(t,u,x): -\delta < t < -\delta, -\infty < u < \infty\ , 0 < x < \infty\}$ by the constant M (say). Thus, for given $\epsilon > 0$, we can choose δ so small that,

$$\left|I_{1,r}(u,x)\right| = \frac{1}{\pi}\left|\int_{-\delta}^{\delta} H(t,u,x)\ \sin rt\ \ dt\right| < \frac{2M\delta}{\pi} < \epsilon$$

for $-\infty < u < \infty$ and $0 < x < \infty$. Fix δ in this way.

Next consider$I_{1,r}(u,x)$: set

$I_{1,r}(u,x) = J_{1,r}(u,x) + J_{2,r}(u,x)$

where,

$$J_{1,r}(u,x) = \frac{1}{\pi}\int_{-\infty}^{\delta} u^k(1+x)^l D_u^q (xD_x)^h\phi(t+u,x)\ \frac{\sin rt}{t}\ dt$$

$J_{2,r}(u,x) = \frac{1}{\pi} u^k (1+x)^l D_u^q (xD_x)^h \phi(u,x) \int_{-\infty}^{r\delta} \frac{\sin z}{z}\ dz$

since $u^k(1+x)^l D_u^q (xD_x)^h \phi(u,x)$ is continuous and bounded support, it is bounded on $-\infty < u < \infty$. By the convergence of the improper integral $\int_{-\infty}^{0} \frac{\sin z}{z} dz$ it follows that $J_{2,r}(u,x)$ tends uniformly to zero on $-\infty < u < \infty, 0 < x < \infty$ as $r \to \infty$.

To show that $J_{1,r}(u,x)$ does the same, first integrate by parts and use the fact that $\phi(u,x)$ is of bounded support to obtain,

$$J_{1,r}(u,x) = \frac{1}{\pi} u^k (1+x)^l D_u^q (xD_x)^h \phi(u+\delta,x) \frac{\cos r\delta}{\delta} +$$

$$\frac{1}{\pi r} \int_{-\infty}^{\delta} u^k (1+x)^l \cos rt\, D_t [\tfrac{1}{t} D_u^q (xD_x)^h \phi(t+u,x)]\ \ dt.$$

The first term on the right hand side tends uniformly to zero on $-\infty < u < \infty$ and $0 < x < \infty$ and $r \to \infty$ because δ is fixed and $u^k(1+x)^l D_u^q (xD_x)^h \phi(u+\delta,x)$ is bounded function of u. Moreover by using the product rule of differentiation with respect to t for equation (1.3.3),

$$u^k(1+x)^l D_t \left[\tfrac{1}{t} D_u^q (xD_x)^h \phi(t+u,x)\right] = u^k(1+x)^l D_u^q (xD_x)^h \phi(t+u,x) \left(\tfrac{-1}{t^2}\right)$$

$$+u^k(1+x)^l \tfrac{1}{t} (xD_x)^h\ \ D_u^{q+1} \phi(t+u,x) \qquad (1.3.4)$$

But for every k, $u^k(1+x)^l D_u^q (xD_x)^h \phi(t+u,x)$ is bounded. This is because $D_u^q (xD_x)^h \phi(t+u,x)$ is bounded and has its support contained in the strip $\{(t,u,x)\colon |t,u,x| < A\}$, where A is sufficiently large number, whereas $u^k(1+x)^l$ is a bounded on the strip. Thus equation (1.3.4) is bounded on the domain. $\{(t,u,x)\colon -\infty < t < -\delta, -\infty < u < \infty\ , 0 < x < \infty\}$, by N (say). This result and assumption that the support of $\phi(u,x)$ is contained in the interval $-A < u < A$ implies that the second term on the right hand side of equation (1.3.4) is bounded by $\frac{2NA}{\pi}$, which tends to zero as $r \to \infty$. Therefore, $I_{1,r}(u,x)$ tends uniformly to zero on $-\infty < u < \infty$ and $0 < x < \infty$ as $r \to \infty$. A similar argument shown that $I_{3,r}(u,x)$ also tends uniformly to zero on $-\infty < u < \infty$ and $0 < x < \infty$ as $r \to \infty$. Thus we have established that $\lim_{r\to\infty} |B_r(u,x)| \le \epsilon$ and since $\epsilon > 0$ is arbitrary, implying that $B_r(u,x)$ converges uniformly to zero, for every $k = 0,1,2,3 \ldots \ldots \ldots$, $-\infty < u < \infty$ and $0 < x < \infty$, by using equation (1.3.1).

1.4 Lemma

For $\phi(t,x) \in \mathcal{D}(\Omega)$, set $G(u,v)$ as in lemma 1.1 then,

$$\frac{i}{4\pi^2} \int_{-r}^{r} \int_{0}^{\tau} \psi(s,y)\ \ \frac{pe^{-isu}}{(v+y)^{p+1}} ds\ \ dw, \qquad (1.4.1)$$

converges in $FS_\alpha(\Omega)$ to $\phi(u,v)$, as $r,\tau \to \infty$.

Proof: Set $G(u,v)$ as in lemma 1.1. In lemma 3.6.1 we have proved that $\gamma_{k,l,q,h} G(u,v) < \infty$, hence $G(u,v)$ converges in $FS_\alpha(\Omega)$. Therefore,
$\gamma_{k,l,q,h} |(4\pi^2)^{-1} i\ G| < \infty$ and $(4\pi^2)^{-1} i\ G$ converges in $FS_\alpha(\Omega)$. Hence to prove the lemma we have to prove that equation (1.4.1) converges to $\phi(u,v)$. Equation (1.4.1) can written by using equation (1.4.1) as,

$$\frac{i}{4\pi^2} \int_{-r}^{r} \int_{0}^{\tau} \frac{pe^{-isu}}{(v+y)^{p+1}} \left[\int_{-\infty}^{\infty} \int_{0}^{\infty} \phi(t,x) e^{ist} (x+y)^{p-1}\ \ dt\ \ dx\right]\ \ ds\ \ dw$$

The order of integration for the repeated integral herein may be changed because $\phi(t,x)$ is of bounded support and the integrand is continuous function of (t,x,s,w) and then by using lemma 1.2 the integral reduces to

$$\frac{-i}{2\pi^2}\int_{-\infty}^{\infty}\int_{0}^{\infty}\phi(t+u,x)\ \frac{\sin rt}{t}\frac{1}{(x-v)}dx\ \ dt$$

By changing the order of integration, we get

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\left[\pi^{-1}\int_{0}^{\infty}\phi(t+u,x)\frac{\sin rt}{t}dt\right]\ \frac{1}{(x-v)}dx$$

In view of lemma 1.3, the term in the bracket converges in $FS_\alpha(\Omega)$ to $\phi(u,x)$ as $r\to\infty$. Therefore, above equation tends to

$$\frac{1}{2\pi i}\int_{0}^{\infty}\phi(u,x)\ \frac{1}{(x-v)}dx$$

Now, by using well known Cauchy integral formula, the last expression tends to $\phi(u,v)$ as $v\to\infty$.

We are now ready to prove inversion theorem.

2. Inversion theorem

Let $f(t,x)$ be a Fourier-Stieltjes transformable generalized function and $F(s,y)$ the distributional Fourier-Stieltjes transform of $f(t,x)$. Then in sense of convergence in $\mathcal{D}'(\Omega)$,

$$f(t,x)=\lim_{r,v\to\infty}\frac{1}{4\pi^2}\ \int_{-s}^{s}\int_{0}^{\rho+iw}F_2(s,y)e^{ist}\,(x+y)^{p-1}\ dy\ ds$$

where, s and ρ are any fixed real number such that $\sigma_1<s<\sigma_2,\rho_1<\rho<\rho_2$ and F_2 is the partial derivative of $F(s,y)$ with respect to y.

Proof: The idea of this proof is to transfer the inversion formula onto a transform of $\phi(t,x)\in\mathcal{D}(\Omega)$ and to use the fact that the resulting expression converges to ϕ with respect to the topology of the testing function space $FS_\alpha(\Omega)$.

Let $\phi(t,x)\in\mathcal{D}(\Omega)$ and we have to show that,

$$\lim_{r,v\to\infty}\langle\frac{1}{4\pi^2}\int_{-s}^{s}\int_{0}^{\rho+iw}F_2(s,y)e^{ist}\,(x+y)^{p-1}\ ds\ dy,\ \phi(t,x)\rangle=\langle f,\phi\rangle$$

From the analyticity of $F(s,y)$ on Ω, the above integral is continuous function of (t,x). Therefore the left hand side without the notation can be written as ,

$$\int_{-\infty}^{\infty}\int_{0}^{\infty}\phi(t,x)\left[\frac{1}{4\pi^2}\int_{-s}^{s}\int_{0}^{\rho+iw}F_2(s,y)e^{ist}\,(x+y)^{p-1}\ ds\ dy\right]\ dx\ dt$$

Considering $y=\rho+iw$ we get

$$\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{0}^{\infty}\phi(t,x)\int_{-r}^{r}\int_{0}^{\tau}F_2(s,y)e^{ist}\,(x+y)^{p-1}\ ds\ dw\,dx\ dt$$

since $\phi(t,x)$ has a compact support and integrand is a continuous function of (t,x,s,w), the order of integration may be changed to yield,

$$\frac{1}{4\pi^2}\int_{-r}^{r}\int_{0}^{\tau}F_2(s,y)\int_{-\infty}^{\infty}\int_{0}^{\infty}\phi(t,x)e^{ist}\,(x+y)^{p-1}\ dx\ dt\,ds\ dw$$

$$=\frac{1}{4\pi^2}\int_{-r}^{r}\int_{0}^{\tau}\langle f(u,v),\frac{pe^{-isu}}{(v+y)^{p+1}}\rangle\int_{-\infty}^{\infty}\int_{0}^{\infty}\phi(t,x)e^{ist}\,(x+y)^{p-1}\ dx\ dt\,ds\ dw$$

$$=\frac{1}{4\pi^2}\int_{-r}^{r}\int_{0}^{\tau}\langle f(u,v),\frac{pe^{-isu}}{(v+y)^{p+1}}\rangle\,\psi(s,y)\,ds\ dw,\qquad \text{\{by using the equation (1.1.1)\}}$$

$= \langle f(u,v), \frac{1}{4\pi^2}\int_{-r}^{r}\int_0^{\tau}\psi(s,y)\frac{pe^{-isu}}{(v+y)^{p+1}}ds\ \ dw\rangle,$ {by using the lemma 1.1}

$= \langle f(u,v), \phi(u,v)\rangle,$ {by using lemma 1.4}

as $r,\tau \to \infty$, which complete the proof of the inversion theorem.

3. Uniqueness theorem

If $FS\{f(t,x)\} = F(s,y)$ for $s,y \in \Omega_f$ and $FS\{f(t,x)\} = G(s,y)$ for $s,y \in \Omega_g$, for $s > 0, \rho_1 < Re\ \ y < \rho_2$. If $\Omega_f \cap \Omega_g$ is not empty and if$F(s,y) = G(s,y)$, for s and $y \in \Omega_f \cap \Omega_g$ then $f = g$ in the sense of equality in $\mathcal{D}^*(\Omega)$.

Proof: f and g must assign the same value to each $\phi \in \mathcal{D}$. By inversion theorem and equating $F(s,y)$ and $G(s,y)$in

$\langle f(t,x) - g(t,x), \phi(t,x)\rangle$

$$= \lim_{\substack{r\to\infty \\ \tau\to\infty}} \langle \frac{1}{4\pi^2}\int_{-r}^{r}\int_0^{\tau}\{F(s,y) - G(s,y)\}e^{ist}(x+y)^{p-1}\ dy\ \ ds, \phi(t,x)\rangle = 0$$

Thus, $f = g$ in $\mathcal{D}^*(\Omega)$.

Hence the theorem.

CONCLUSION

In the given work we have proved inversion theorem for the two dimensional Fourier-Mellin transform. So we have extended the two dimensional Fourier-Mellin transform and generalized transform in conventional way.

REFERENCE

[1]David Boteler(2012): On Choosing Fourier Transforms for Practical Geoscience Applications, International Journal of Geosciences, 3, 952-959.

[2]Debnath Loknath(2007): Integral Transform and their application, Second edition, London, New York.

[3]Deshna Loonker(2016): Solution Of Integral Equations And Laplace -Stieltjes Transform, Palestine Journal of Mathematics, Vol. 5(1), 43–49 .

[4]Dennis Nemzer(2014): Extending The Stieltjes Transform, Sarajevo Journal Of Mathematics, Vol.10 (23), 197{208).

[5]Janelle K. Hammond(2011): Mathematics of Music, UW-L Journal of Undergraduate Research XIV.

[6]Mih´Aly Kov´Acs And Frank Neubrander: On The Inverse Laplace-Stieltjes Transform Of A-Stable Rational Functions,2000 Mathematics Subject Classification. Primary: 44A10. Secondary: 26C15.

[7]R. S. Pathak, Lokenath Debnath(1987): Recent Developments On The Stieltjes Transform Of Generalized Functions, Internat. J. Math. & Math. Sci.,Vol. i0 No. 4, 641-670.

[8]S.S.Pandey, Manu Pratap Singh and Vikas Pandey((April 2015): Image Transformation and Compression using Fourier Transformation, International Journal of Current Engineering and Technology, Vol.5, No.2.

[9]Sharma V.D. and Dolas P.D.(2012): Analyticity of Distribution Generalized FS transform, Int. Jr. of Math. Analysis,Vol. 6, No. 9, 447-451.

[10]V. D. Sharma, P. D. Dolas(December 2011): Inversion Formula For Two Dimensional Generalized Fourier-Mellin Transform And It's Application, Int. Jr. of Advanced Scientific and Technical Research, Issue 1, Vol. 2.