ROUGH PRIME IDEALS OF NEAR-RINGS

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Abstract

This paper is an extension work of the paper by N.Thillaigovindan and V.S.Subha[13]. In this paper the notion of congruence relation in a near-ring and the lower and upper upproximations of an ideal with respect to the congruence relation is introduced. Also the notion of prime ideals in a near-ring is introduced and prime ideals is characterized in terms of rough approximations.

Keywords: Near-rings, Ideals; Prime ideals Lower approximation; Upper approximation.

1 INTRODUCTION

The notion near-ring is introduced by Pilz[10]. Berkenmeier et. al.,[1] developed prime ideals in near-rings. Pawlak introduced the notion of rough set in his papers[7-9]. Rough set theory, a new mathematical approach to deal with in exact, uncertain or vague. Knowledge has recently received wide attention on the research areas in both of the real-life applications and theory itself. It has found practical applications in many areas such as knowledge discovery machine learning, data analysis, approximate classification, conflict analysis and so on. The algebraic approach of rough set was studied by some others such as[2,3,4,5,11,12,14]. Thillaigovindan and Subha[13] introduced rough ideals in near-rings. Osman and Davvaz[6],studied rough prime (primary) ideals and rough fuzzy prime (primary) ideals in commutative rings.

In this paper the notion of congruence relation in a near-ring and the lower and upper upproximations of an ideal with respect to the congruence relation is introduced. Also the notion of prime ideals in a near-ring is introduced and prime ideals is characterized in terms of rough approximations.

2 PRELIMINARIES

In this section, we give some definitions and introduce the necessary notations which will be used throughout this paper.

Let U be a universal set. for an equivalence relation θ on U, the set of elements of U that are related to $x \in U$, is called the equivalence class of x and is denoted by $[x]_\theta$. Let U/θ denote the family of all equivalence classes induced on U by θ. U/θ is a partition or classification of U such that each element of U is contained in exactly on element of U/θ.

Definition 2.1. [13] A pair (U,θ) where $U \neq \emptyset$ and θ is an equivalence relation on U, is called an approximation space.

Definition 2.2. [13] For an approximation space (U,θ) by a rough approximation in (U,θ) we mean a mapping $\rho: \wp(U) \to \wp(U) \times \wp(U)$ defined as $\rho(X) = (\underline{\rho}(X), \overline{\rho}(X))$ for $X \subseteq U$, where $\underline{\rho}(X) = \{x \in X | [x]_\rho \subseteq X\}$ and $\overline{\rho}(X) = \{x \in X | [x]_\rho \cap X \neq \emptyset\}$, $\underline{\rho}(X)$ is called lower rough approximation of X in (U,θ), where $\overline{\rho}(X)$ is called upper approximation of X in (U,θ).

Definition 2.3.[13] Given an approximation space (U,θ) a pair $(A,B) \in \wp(U) \times \wp(U)$ is caled rough set in (U,θ) if and only if $(A,B) \in \rho(X)$ for some $X \subseteq U$.

Throughout this chapter N denotes a near-ring unless otherwise mentioned. A near-ring is a non-empty set N together with two binary operations + and · such that:

i. $(N,+)$ is a group (not necessarily abelian)
ii. $(N,\cdot)$ is a semigroup
iii. $(n_1 + n_2) \cdot n_3 = n_1 \cdot n_3 + n_2 \cdot n_3$, for all $n_1, n_2, n_3 \in N$.

In near-ring, $0 \cdot x = 0$ and $(-x) \cdot y = -x \cdot y$, but in general $x \cdot 0 \neq 0$ for some $x \in N$.

Recall that an equivalence relation θ on N is a reflexive, symmetric and transitive binary relation on N. If θ is an equivalence relation on N, then the equivalence class of $x \in N$ is the set $\{y \in N | (x,y) \in \theta\}$ and is denoted by $[x]_\theta$. If A and B are two subsets of N then $AB = \{ab | a \in A, b \in B\}$.

Definition 2.4. Let θ be an equivalence relation on N. θ is called a congruence relation if $(a,b) \in \theta$ implies $(a+x, b+x)$, (ax, bx) and $(xa, xb) \in \theta$ for all $x \in N$.

Theorem 2.5. Let θ be a congruence relation N. Then $(a,b),(c,d) \in \theta$ implies $(a+c, b+d)$, (ac, bd) and $(-a,-b) \in \theta$ for all $a,b,c,d \in N$.

Proof. It is easily obtained by applying the Definition 2.4.

Lemma 2.6. Let θ be a congruence relation N. If $a, b \in N$, then

(i) $[a]_\theta + [b]_\theta = [a+b]_\theta$
(ii) $[-a]_\theta = -[a]_\theta$
(iii) $[a]_\theta [b]_\theta \subseteq [ab]_\theta$

Proof. (i) Let $x \in N$. Suppose $x \in [a]_\theta + [b]_\theta$. Then there exist $y, z \in N$ such that $y \in [a]_\theta, z \in [b]_\theta$ and $x = y + z$. This means that $(a,y),(b,z) \in \theta$ and hence $(a+b, y+z) = (a+b, x) \in \theta$. Thus $x \in [a+b]_\theta$ and therefore $[a]_\theta + [b]_\theta \subseteq [a+b]_\theta$.

Again suppose that $y \in [a+b]_\theta$. Then $(a+b, y) \in \theta, (a, y-b) \in \theta$ and so $y - b \in [a]_\theta$. This implies that $y \in [a]_\theta + b \subseteq [a]_\theta + [b]_\theta$. Thus $[a+b]_\theta \subseteq [a]_\theta + [b]_\theta$. Combining the two results we have $[a+b]_\theta = [a]_\theta + [b]_\theta$.

(ii) Let $x \in [-a]_\theta$. Then $(-a, x) \in \theta$. By Theorem 2.5, we have $(a, -x) \in \theta$. Thus $-x \in [a]_\theta$ and $x \in [-a]_\theta$. Again assume that $y \in -[a]_\theta$. This implies that $-y \in [a]_\theta$ and $(a, -y) \in \theta$. Thus $y \in [-a]_\theta$.

(iii) Let $z = xy \in [a]_\theta[b]_\theta$. Then $x \in [a]_\theta$ and $y \in [a]_\theta$. This implies that $(a, x) \in \theta$ and $(b, y) \in \theta$. Thus $z = xy \in [ab]_\theta$.

A congruence relation θ on N is called complete if $[ab]_\theta = [a]_\theta[b]_\theta$.

Definition 2.7 Let θ be a congruence relation on N and A a subset of N. Then the sets

$\underline{\theta}(A) = \{x \in N / [a]_\theta \subseteq A\}$ *and* $\overline{\theta}(A) = \{x \in N / [a]_\theta \cap A \neq \emptyset\}$ are called the *lower and upper approximations* of the set A, respectively.

Let A be any subset of N. $\theta(A) = (\underline{\theta}(A), \overline{\theta}(A))$ is called a *rough set* with respect to θ if $\underline{\theta}(A) \neq \overline{\theta}(A)$.

Definition 2.8. Let $(N, +, \cdot)$ be a near-ring. A subset I of N is called an ideal of N if

1) $(I, +)$ is anormal subgroup of $(N, +)$
2) $IN \subseteq I$
3) $n_1 \cdot (n_2 + i) - n_1 \cdot n_2 \in I \in N$.

For all $i \in I$ and $n_1, n_2 \in N$.

If I satisfies (1) and (2) then it is called a *right ideal* of N. If I satisfies (1) and (3), then it is called a *left ideal* of N.

Definition 2.9. Let A be any subset of N and (N, θ) be a rough approximation space. If $\underline{\theta}(A)$ *and* $\overline{\theta}(A)$ are ideals, then $\underline{\theta}(A)$ is called a *lower rough ideal* and $\overline{\theta}(A)$ is called a *upper rough ideal* of N, respectively. $\theta(A) = (\underline{\theta}(A), \overline{\theta}(A))$ is called a *rough ideal.*

Theorem 2.10. *Let θ be a congruence relation on N. If I is an ideal of N, then $\overline{\theta}(I)$ is an ideal of N.*

Proof. Let $a, b \in \overline{\theta}(I)$. then $[a]_\theta \cap I \neq \emptyset$ and $[b]_\theta \cap I \neq \emptyset$. So there exist $x \in [a]_\theta \cap I$ and $y \in [b]_\theta \cap I$. Since $x, y \in I, x - y \in I$. Now $x - y \in [a]_\theta - [b]_\theta = [a - b]_\theta$.

Therefore $[a - b]_\theta \cap I \neq \emptyset$ and this means that $a - b \in \overline{\theta}(I)$. Let $i \in \overline{\theta}(I)$ and $n \in N$.

Then there exists $x \in [i]_\theta \cap I$ and since I is an ideal of $N, n + x - n \in I$ and

$n + x - n \in [n]_\theta + [i]_\theta - [n]_\theta = [n + i - n]_\theta$. Thus $[n + i - n]_\theta \cap I \neq \emptyset$. This implies that

$n + x - n \in \overline{\theta}(I)$.

Assume that $i \in \overline{\theta}(I)$ and $n \in N$. Then there exists $x \in [i]_\theta \cap I$ and $(x, i) \in \theta$. Since θ is a congruence relation, $(xn, in) \in \theta$. This implies that $xn \in [in]_\theta$. Hence $xn \in [in]_\theta$ and we have $xn \in \overline{\theta}(I)$. Again let $i \in \overline{\theta}(I)$ and $n, n' \in N$. This means that there exists $y \in [i]_\theta \cap I$ and $n, n' \in I$ such that $n(n' + y) - nn' \in I$. Since $y \in [i]_\theta, (y, i) \in \theta$ and $n(n' + y) - nn', n(n' + i) - nn') \in \theta$ implying that $n(n' + y) - nn' \in [n(n' + i) - nn']_\theta$. Thus $[n(n' + i) - nn'] \in \overline{\theta}(I)$ and hence $\overline{\theta}(I)$ is an ideal of N.

Remark 2.11. As the condition of the theorem is only necessary, the question of converse does not arise.

Theorem 2.12. *Let θ be a congruence relation on N and I be an ideal. If $\underline{\theta}(A)$ is a nonempty set, then it is equal to I.*

Proof. Since $\underline{\theta}(A)$ is non-empty, there exists $x \in N$ such that $x \in \underline{\theta}(I)$. Since $x \in \underline{\theta}(I). x \in [x]_\theta \subseteq I$. Suppose that $a \in I$.

$$\begin{aligned}[0]_\theta &= [x + (-x)]_\theta \\ &= [x]_\theta + [-x]_\theta \\ &= [x]_\theta + ([-x]_\theta) \\ &\subseteq I + I \\ &\subseteq I.\end{aligned}$$

Since I is an ideal of N. We have $a + [0]_\theta \subseteq a + I \subseteq I$. As

$$\begin{aligned}x \in a + [0]_\theta &\Leftrightarrow x - a \in [0]_\theta \\ &\Leftrightarrow (x - a, 0) \in \theta \\ &\Leftrightarrow (x, a) \in \theta \\ &\Leftrightarrow x \in [a]_\theta, [a]_\theta \in I.\end{aligned}$$

Hence $a \in \underline{\theta}(I)$. Thus $\underline{\theta}(I) = I$.

Corollary 2.13. If I is an ideal of N and $\underline{\theta}(I)$ is nonempty, then $\theta(A) = (\underline{\theta}(A), \overline{\theta}(A))$ is a rough ideal of N.

Lemma 2.14. If I and J is an ideal of N and $\underline{\theta}(I \cap J)$ is a nonempty set, then $(\underline{\theta}(I \cap J), \overline{\theta}(I \cap J))$ is a rough ideal of N.

Proof. Since the intersection of two ideals of N is an ideal of N, $I \cap J$ is an ideal of N. By Theorem 2.10, $\overline{\theta}(I \cap J)$ is an ideal of N. Also by Theorem 2.12 $\underline{\theta}(I \cap J)$ is an ideal of N.

Theorem 2.11. Let φ be an epimorphism of a near-ring N_1 to a near-ring N_2 and let θ_2 be a congruence relation on N_2. Then

i) $\theta_1 = \{(a, b) \in N_1 \times N_1 / (\varphi(a), \varphi(b)) \in \theta_2\}$ is a congruence relation
ii) If θ_2 is complete and φ is $1 - 1$, then $\theta_1$is complete$\varphi(\overline{\theta}_1(A)) = \overline{\theta}_2(\varphi(A))$
iii) $\varphi\big(\underline{\theta}_1(A)\big) \subseteq \underline{\theta}_2\big(\varphi(A)\big)$
iv) If φ is $1 - 1$, then $\varphi\big(\underline{\theta}_1(A)\big) = \underline{\theta}_2\big(\varphi(A)\big)$.

Proof. (i) Let $a, b \in \theta_1$ and $x \in N_1$. Then $(\varphi(a), \varphi(b)) \in \theta_2$. Since θ_2 is a congruence relation $(\varphi(a), +\varphi(b), (\varphi(b) + \varphi(x) \in \theta_2, (\varphi(a)\, \varphi(x), (\varphi(b)\, \varphi(x)) \in \theta_2$. Again since φ is homomorphism, $\varphi(a + x), \varphi(b + x), \varphi(ax), \varphi(bx) \in \theta_2$ and φ being onto, $(a + x, b + x) \in \theta_1$ and $(ax, bx) \in \theta_1$.

(ii) Let $z \in [ab]_{\theta_1}$. Then $(ab, z) \in \theta_1$. By definition of $\theta_2, \varphi(ab), \varphi(z) \in \theta_2$. Hence

$\varphi(z) \in [\varphi(ab)]_{\theta_1}.[\varphi(b)]_{\theta_2}$, there exist $x, y \in N_1$ such that

$$\varphi(z) = \varphi(x)\varphi(y)$$
$$= \varphi(xy), \varphi(x) \in [\varphi(a)]_{\theta_2}, \varphi(y) \in [\varphi(b)]_{\theta_2}.$$

Since φ is $1-1$ and by definition of $\theta_1, z = xy$ and $x \in [a]_{\theta_1}, y \in [b]_{\theta_1}$. Thus $z \in [a]_{\theta_1}.[b]_{\theta_1}$ and therefore $[ab]_{\theta_1} \subseteq [a]_{\theta_1}[b]_{\theta_1}$. By Lemma 4.5.3(iii), $[a]_{\theta_1}[b]_{\theta_1} \subseteq [ab]_{\theta_1}$. Hence θ_1 is complete.

(iii) Let $y \in \varphi\left(\overline{\theta}_1(A)\right)$. Then there exists $x \in \overline{\theta}_1(A)$ such that $y \in \varphi(x)$. This implies that $[x]_{\theta_1} \cap A \neq \emptyset$ and so there exists $a \in [x]_{\theta_1} \cap A$. Then $\varphi(a) \in \varphi(A)$ and $(a, x) \in \theta_1$ implies $\left(\varphi(a), \varphi(x)\right) \in \theta_2$. So $\varphi(a) \in [\varphi(x)]_{\theta_2}$. Thus $[\varphi(x)]_{\theta_2} \cap \varphi(A) \neq \emptyset$. This implies that

$y = \varphi(x) \in \overline{\theta}_1(\varphi(A))$ and so $\varphi\left(\overline{\theta}_1(A)\right) \subseteq \overline{\theta}_2(\varphi(A))$. (1)

Again let $z \in \overline{\theta}_2(\varphi(A))$, then there exists $z \in N_1$ such that $z = \varphi(x)$.

Hence $[\varphi(x)]_{\theta_2} \cap \varphi(A) \neq \emptyset$. So there exists $a \in A$ such that $\varphi(a) \in \varphi(A)$ and $\varphi(a) \in [\varphi(x)]_{\theta_2}$.

By definition of θ_1, we have $a \in [x]_{\theta_1}$. Thus $[x]_{\theta_1} \cap A \neq \emptyset$, which implies $x \in \overline{\theta}_1(A)$ so $z = \varphi(x) \in \varphi(\overline{\theta}_1(A))$. It means that $\overline{\theta}_2(\varphi(A)) \subseteq \varphi(\overline{\theta}_1(A))$. (2)

From (1) and (2) the conclusion follows.

(iv) Let $y \in \varphi\left(\underline{\theta_1}(A)\right)$. Then there exists $x \in \underline{\theta_1}(A)$ such that $\varphi(x) = y$ and so we have $[x]_{\theta_1} \subseteq A$. Again let $b \in [y]_{\theta_2}$. Then there exists $a \in N_1$ and such that $\varphi(a) = b$ and $\varphi(a) \in [\varphi(x)]_{\theta_2}$. Hence $a \in [x]_{\theta_1} \subseteq A$ and so $b = \varphi(a) \in \varphi(A)$. Thus $[y]_{\theta_2} \subseteq \varphi(A)$. This implies that $y \in \underline{\theta_2}(\varphi(A))$. And so we have $\varphi(\underline{\theta_1}(\varphi(A)) \subseteq \underline{\theta_2}(\varphi(A))$.

(v) Let $y \in \underline{\theta_2}(\varphi(A))$. Then there exists $x \in N_1$ such that $\varphi(x) = y$ and $[\varphi(x)]_{\theta_2} \subseteq \varphi(A)$. Let $a \in [x]_{\theta_1}$. Then $\varphi(a) \in [\varphi(x)]_{\theta_2}$ and so $b \in A$. Thus $[x]_{\theta_1} \subseteq A$ and $x \in \underline{\theta_1}(A)$. Hence $y = \varphi(x) \in \varphi(\underline{\theta_1}(A))$ and so we have $\underline{\theta_2}(\varphi(A)) \subseteq \varphi(\underline{\theta_1}(A))$. By (iv), we have $\varphi(\underline{\theta_1}(A)) = \underline{\theta_2}(\varphi(A))$.

3. ON ROUGH PRIME IDEALS IN NEAR-RINGS

An ideal A in a near-ring N is said to be prime if for any ideals A, B in N such that $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$.

Theorem 3.1. [10] Let P be an ideal of N. The following are equivalent:

a) P is a prime ideal
b) For every ideal I, J in N such that $IJ \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$
c) For every $i, j \in N, i \notin P$ and $j \notin P$ implies $(i)(j) \nsubseteq P$
d) For every ideal I, J in $N, I \supseteq P$ and $J \supseteq P$ implies $IJ \nsubseteq P$
e) For every ideal I, J in $N, I \nsubseteq P$ and $J \nsubseteq P$ implies $IJ \nsubseteq P$.

Theorem 3.2. Let θ be a congruence relation on N and P a prime ideal of N such that $\overline{\theta}(P) \neq N$, then P is an upper rough prime ideal of N.

Proof. Let P be a prime ideal of N. By Theorem 2.9, $\overline{\theta}(P)$ is an ideal of N. Suppose A and B are ideals of N such that $AB \subseteq \overline{\theta}(P)$. Let $A \nsubseteq \overline{\theta}(P)$ and $B \nsubseteq \overline{\theta}(P)$. Then there exists $a \in A$ such that $a \notin \overline{\theta}(P)$ and then there exists $b \in A$ such that $b \notin \overline{\theta}(P)$.

This implies $[a]_\theta \cap P = \emptyset$ and $[b]_\theta \cap P = \emptyset$. Then $a, b \notin P$. By Theorem 3.1(c), $ab \notin P$ which gives a contradiction to $AB \subseteq \overline{\theta}(P)$. Thus either $A \subseteq \overline{\theta}(P)$ or $B \subseteq \overline{\theta}(P)$ and so $\overline{\theta}(P)$ is a prime ideal of N.

Theorem 3.3. Let θ be a congruence relation on N and P a prime ideal of N. If $\underline{\theta}(P)$ is a nonempty set, then $\underline{\theta}(P)$ is a prime ideal of N.

Proof. By Theorem 2.10, $\underline{\theta}(P) = P$ and so $\underline{\theta}(P)$ is a prime ideal of N.

Corollary 3.4. Let θ be a congruence relation on N and P a prime ideal of N. If $\underline{\theta}(P)$ is a nonempty subset of N, then $(\underline{\theta}(P), \overline{\theta}(P))$ is a rough prime ideal of N.

Remark 3.5. The condition on Theorem 3.3 and Corollary 3.4 are only necessary. Hence the question of converse does not arise.

5.CONCLUSION

In this paper to establish a relationship between rough set theory and near-rings. it is the extension work of rough ideals in near-rings. So the further work will focus the properties of rough prime ideals in near-rings and extended to Γ-near- rings and modules.

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