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APPLICATION OF RESIDUE THEORY FOR A - ANALYTIC FUNCTION

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ABSTRACT

The theory of A-analytic functions is one of the young and promising areas of modern mathematics. The study of A-analytic functions has its own characteristics and difficulties. At present, A-analytic functions, including the concept of deduction for A-analytic functions, are widely used in computational work.

This article explores and analyzes the concept of deduction for A-analytic functions, as well as its application. Analysis of the literature on the research topic showed that studies of A-analytic functions were significantly developed in the works of Arbuzov E.V., Bukhgeim A.L., Sadullaev A., Jabborov N.M. etc., whose work served as the theoretical and practical basis for this work. The topic under consideration is very relevant in many areas of mathematical analysis, and will be of interest to many specialists in this area. The issue of applying the theory of deductions requires further study. The article summarizes the practical experience of applying residue theory for A-analytic functions.

Keywords:*A* – analytic function, *A*-lemniscate residue for *A*-analytic function.

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ПРИМЕНЕНИЕ ТЕОРИИ ВЫЧЕТОВ ДЛЯ А-АНАЛИТИЧЕСКОЙ ФУНКИИ

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АННОТАЦИЯ

Теория А-аналитических функции - одна из молодых и перспективных направлений современной математики. При изучении А-аналитических функции имеются свои особенности и трудности. В настоящее время А-аналитические функции, в том числе понятие вычета для А-аналитических функции широко используется в вычислительных работах.

В данной статье исследуется и анализируется понятие вычета для А-аналитических функций, а также его применения.Анализ литературы по теме исследования показал, что исследования А-аналитических функции получили значительное развитие в работах ArbuzovE.V., BukhgeimA.L.,SadullaevA., JabborovN.M. и т.д., чьи работы послужили теоретической и практической основой данной работы. Рассматриваемая тема очень актуальна во многих областях математического анализа, и будет интересна многим специалистам данного направления. Вопрос применения теории вычетов требует дальнейшего исследования. Статья

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обобщает практический опыт применения теории вычетов для A-аналитических функций.

Ключевые слова: A-аналитическая функция, A-лемниската, вычет для A-аналитической функции.

Let A be an anti-analytic function, i.e. $\partial A = 0,$ in the area $D \subset \mathbb{C}$ and such that $\left|A(z)\right| \le C < 1, \ \forall z \in D.$ Put $D_A = \frac{\partial}{\partial z} - \overline{A}(z)\frac{\partial}{\partial \overline{z}}, \ \ \overline{D}_A = \frac{\partial}{\partial \overline{z}} - A(z)\frac{\partial}{\partial z}$.

Definition 1. The function $f \in C^1(D)$ is called A-analytic in D, if for all $z \in (D)$

$$\overline{D}_A f(z) = 0.$$

Since the anti-analytic function is infinitely smooth, it follows from (see [1]) that $O_A(D) \subset C^\infty(D)$. Since the domain is D-simply connected and $\overline{A}(z) -$ holomorphic function, the integral $I(z) = \int\limits_{\gamma(\xi,z)} \overline{A}(\tau)\,d\tau$ is independent of the integration path; it coincides with the antiderivative, $I'(z) = \overline{A}(z)$. The function $\psi(z,\xi) := z - \xi + \overline{\int\limits_{\gamma(\xi,z)} \overline{A}(\tau)\,d\tau}$ is A-analytic in the area D, has a single prime zero at a point $z = \xi$. It implements an internal mapping. In particular, the set $L(\xi,r) = \left\{ z \in D: \ \left|\psi(z,\xi)\right| = \left| z - \xi + \overline{\int\limits_{\gamma(\xi,z)} \overline{A}(\tau)\,d\tau} \right| < r \right\}, \ r > 0$ is an open set in D. For sufficiently small r it compactly belongs to D and contains the point ξ

. This set is called the A-lemniscate, centered at the point ξ and is denoted by $L(\xi, r)$. *A*-lemniscate is a simply connected region (see [2]).

Definition 2.A point $a \in C$ is called an isolated singular point of the function $f(z)$ if there exists a punctured neighborhood of the lemniscate of this point (i.e., the set $0 < |\psi(z,a)| < r$), if the point a is finite, or a set $R < \left| z + \int_0^z \overline{A(\tau)} d\tau \right| < \infty$, $A \equiv const.$, $|A| < 1$ if $a = \infty$, in which the function $f(z)$- is *A*-analytic.

Definition 3. An isolated singular point a of the function $f(z)$ is called a removable point if there exists a finite $\lim_{z \to a} f(z) = B$ pole, if there exists a $\lim_{z \to a} f(z) = \infty$ substantially singular point, if $f(z)$ has neither a finite nor an infinite limit for $z \to a$.

Theorem 1. (see. [2,3]). For the function $f(z)$- *A*-analytic in the closed constraints of the multiply connected domain *D*, then the integral over the measure $d\xi + A(\xi) d\bar{\xi}$ taken over the entire boundary of this region, passed so that the region remains on the left, is zero i.e.

$$\oint_{\partial G} f(\xi)(d\xi + A(\xi) d\bar{\xi}) = 0.$$

Theorem 2. (expansion in the Laurent series, see [3]). Let $f(z)$ be *A*-analytic in the ring of lemniscate: $f(z) \in O_A(L(a,R) \setminus L(a,r))$, $r < R$. Then in this ring $f(z)$ decomposes in the Laurent series:

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$$f(z)=\sum_{k=0}^{\infty} c_k \psi^k(z,a), \tag{1}$$

where the coefficients of the series are determined by the formula

$$c_k=\frac{1}{2\pi i}\int_{\partial L(a,\rho)} \frac{f(\xi)}{\left[\psi(\xi,a)\right]^{k+1}}\left(d\xi+A(\xi)d\bar{\xi}\right),\ r<\rho<R,\ k=0,\pm 1,....$$

Series (1) converges uniformly inside the ring

$$L(a,R)\setminus\overline{L(a,r)}=\left\{z\in D:\ r<\left|\psi(z,a)\right|<R\right\}.$$

Theorem 3. If the point *a* is zero of the *A*-analytic function $f(z)$, which is not identically equal to zero in any neighborhood of $L(a,r)$, then there exists a natural number n such that $f(z)=\left(\overline{z-a+\int_{\gamma(z,a)}\overline{A(\tau)}d\tau}\right)^n\varphi(z)$, where the function $\varphi(z)$ is *A*-analytic at the point *a* and is nonzero in some neighborhood of this point.

An isolated singular point $a\in\Box$ of the function $f(z)$ is removable if and only if the Laurent expansion of $f(z)$ in the neighborhood of *a* does not contain the main part: $f(z)=\sum_{k=0}^{\infty}c_k\left(\overline{z-a+\int_{\gamma(z,a)}\overline{A(\tau)}d\tau}\right)^n$. An isolated singular point $a\in\Box$ of $f(z)$ is a pole if and only if the principal part of the Laurent expansion of $f(z)$ in the neighborhood of *a* contains only a finite (and positive) number of nonzero terms: $f(z)=\sum_{n=-N}^{\infty}c_n\left(\overline{z-a+\int_{\gamma(z,a)}\overline{A(\tau)}d\tau}\right)^n$.

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Let D be a domain that is ∂G-smooth or piecewise-smooth and fess, where $f\in O_A\left(D\setminus\{a_1,...,a_n\}\right)\cap C\left(\overline{D}\right)$ is a finite number of isolated singular points. Let $G_r=\left\{z\in D:\ \inf\limits_{\xi\in\partial D}|z-\xi|>r\right\}\subset D$ and r satisfy the conditions $L\left(a_k,r\right)\cap L\left(a_j,r\right)=\varnothing$ and $\bigcup\limits_{k=1}^{n}L(a_k,r)\subset G_r$..

Then ∂G_r – is an arbitrary piecewise-smooth closed contour containing inside it points $a_1,...,a_n$ and lying entirely in the area*D*. Since the function*f* is *A*-analytic at each point of the closed region, bounded by a complex contour $G_r\setminus\left(\bigcup\limits_{k=1}^{n}L(a_k,r)\right)$, by Cauchy's theorem we have the equality

$$\oint\limits_{\partial G_r} f(\xi)(d\xi+A\left(\xi\right)d\,\overline{\xi})=\sum_{k=1}^{n}\oint\limits_{\partial L(a_k,r)} f(\xi)(d\xi+A\left(\xi\right)d\,\overline{\xi})$$

Definition 4. The integral of the function $f\left(z\right)$ over a sufficiently small *A*-lemniscate $L(a,r)$, divided by *2πi* is called the residue $f\left(z\right)$ is an A -analytic function at the point $z=a$,

$$\underset{z=a}{res}\,f(z):=\frac{1}{2\pi i}\oint\limits_{\partial L(a,r)} f(\xi)(d\xi+A(\xi)d\overline{\xi}) \qquad (2)$$

then the following analogue of the Cauchy theorem on *A*-residues holds.

Theorem 4. Let a function $f\left(z\right)$ be A-analytic everywhere, except for an isolated set of singular points and a domain $G\subset\subset D$ and its boundary ∂G not contain singular points, then

$$\oint_{\partial G_r} f(\xi)(d\xi + A(\xi)d\bar{\xi}) = 2\pi i \sum_{k=1}^{n} \underset{z=a_k}{res_A} f(z)$$

Theorem 5. The residue of the function $f(z)$ at an isolated singular point $a \in D$ is equal to the coefficient for minus the first degree $\psi(z,a)$ in its Laurent expansion in the neighborhood of $z = a$ with the A-lemniscate $L(a,r)$:

$$\underset{z=a}{res_A} f(z) = c_{-1}$$

Evidence. We know that

$$\oint_{\partial L(a,r)} \psi(z,a)^n (d\xi + A(\xi)d\bar{\xi}) = \begin{cases} 2\pi i, n = -1 \\ 0, n \in Z \setminus \{-1\} \end{cases}$$

Then, from equality 2, integrating over the boundary $\partial L(a,r)$ and with respect to the measure $d\xi + A(\xi)d\bar{\xi}$ then we get that

$$\oint_{\partial L(a,r)} f(\xi)(d\xi + A(\xi)d\bar{\xi}) = 2\pi i c_{-1}.$$

Consequence. (see [5]) Let a point $z = a$ be a pole of order n for an $A(z)-$analytic function $f(z)$, then the following formula holds:

$$\underset{z=a}{res} f(z) = \frac{1}{(n-1)!} \lim_{z \to a} \frac{\partial^{n-1}}{\partial z^{n-1}} \left[f(z) \left(z - a + \overline{\int_{\gamma(a,z)} \overline{A(\tau)} d\tau} \right)^n \right].$$

Theorem 6. For the meromorphic function $R(x,y)$ and the set $L(a,r) \subset\subset D$, the equality

$$\int_0^{2\pi} R(\sin t,\cos t)dt = 2\pi \sum_{a_k \in L(a,r)} \underset{z=a_k}{res_A} \frac{R\left(\frac{\psi(z,a)^2 + r^2}{2r\psi(z,a)}, \frac{\psi(z,a)^2 - r^2}{2ir\psi(z,a)}\right)}{\psi(z,a)}.$$

Evidence. Consider the boundary $L(a,r)$. On the boundary of $\partial L(a,r)$, the following equality is true

$$\psi(z,a) = re^{it}, \overline{\psi(z,a)} = re^{-it}, d\psi(z,a) = d\xi + A(\xi)d\overline{\xi} = rie^{it}dt = i\psi(z,a)dt.$$

It follows that

$$dt = \frac{d\psi(z,a)}{i\psi(z,a)}, \cos t = \frac{e^{it}+e^{-it}}{2} = \frac{\frac{\psi(z,a)}{r} + \frac{r}{\psi(z,a)}}{2} = \frac{\psi(z,a)^2 + r^2}{2r\psi(z,a)},$$

$$\sin t = \frac{e^{it}-e^{-it}}{2i} = \frac{\frac{\psi(z,a)}{r} - \frac{r}{\psi(z,a)}}{2i} = \frac{\psi(z,a)^2 - r^2}{2ir\psi(z,a)}$$

By the theorem 4, we have

$$\int_0^{2\pi} R(\sin t,\cos t)dt = \oint_{|\psi(z,a)|=r} \frac{R\left(\frac{\psi(z,a)^2 + r^2}{2r\psi(z,a)}, \frac{\psi(z,a)^2 - r^2}{2ir\psi(z,a)}\right)}{i\psi(z,a)} d\psi(z,a) =$$

$$= 2\pi \sum_{a_k \in L(a,r)} \underset{z=a_k}{res_A} \frac{R\left(\frac{\psi(z,a)^2 + r^2}{2r\psi(z,a)}, \frac{\psi(z,a)^2 - r^2}{2ir\psi(z,a)}\right)}{\psi(z,a)}.$$

The theorem is proved.

After such a change, the above theorem will also be valid for the case $a = \infty$ and $A = const$. This result can be obtained by replacing the variable $z = \frac{1}{\omega}$. If we denote $f(z) = f\left(\frac{1}{\omega}\right) = g(\omega)$, then obviously

$$\lim_{z\to\infty} f(z) = \lim_{\omega\to\infty} g(\omega)$$

and therefore g has at the point $\omega = 0$ the same singularity as f at the point $z = \infty$. For example, in the case of a pole g in $V = \{0 < |\omega + A\overline{\omega}| < r\}$ has decomposition

$$g(\omega) = \frac{c_{-N}}{\left(\omega + A\overline{\omega}\right)^N} + \ldots + \frac{c_{-1}}{\omega + A\overline{\omega}} + c_0 + c_1(\omega + A\overline{\omega}) + c_2(\omega + A\overline{\omega})^2 + \ldots$$

replacing here $\omega = \frac{1}{z}$, in the ring of the lemniscate $V' = \left\{\frac{1}{r} < |z + A\overline{z}| < \infty\right\}$ we obtain the decomposition

$$f(z) = \ldots + \frac{b_{-2}}{\left(z + A\overline{z}\right)^2} + \frac{b_{-1}}{z + A\overline{z}} + b_0 + b_1(z + A\overline{z}) + \ldots + b_N(z + A\overline{z})^N$$

where $b_n = c_{-n}, b_N \neq 0$. We define the deduction at infinity.

Definition 4. Suppose that the function f has at its point $a = \infty$ its isolated singular point. A residue at infinity is a given quantity

$$\underset{z=\infty}{res}\, f(z) := \frac{1}{2\pi i} \oint_{|z + A\overline{z}| = R} f(z)(dz + Ad\overline{z})$$

where the lemniscate has a sufficiently large radius, passing clockwise.

The orientation of the $\partial L = \{z : |z + A\overline{z}| = R\}$ was chosen so that during its bypass the set of $\{z : R < |z + A\overline{z}| < \infty\}$ remained on the left. We write the Laurent decomposition for f in this set:

$$f(z) = \sum_{n=-\infty}^{\infty} b_n (z + A\overline{z})^n$$

Integrating it along $\partial L = \{z : |z + A\overline{z}| = R\}$ we find $\underset{z=\infty}{res_A}\, f(z) = -b_{-1}$.

An analogue of Jordan's lemma. $f:D\to C$ be an A-analytic function in the domain $D=\left\{z\in C:|z+A\bar{z}|>0\right\}$ everywhere, except for the isolated set of singular points, and

$M(R):=\sup\left\{|f(z)|:|z+A\bar{z}|=R, \operatorname{Im}(z+A\bar{z})\ge 0\right\}, \lim_{R\to\infty} M(R)=0$. Then for any $\lambda>0$ we have the equality

$$\lim_{R\to\infty}\int_{\gamma R} f(z)e^{i\lambda(z+A\bar{z})}(dz+Ad\bar{z})=0$$

where $\gamma_R=\left\{z\in C:|z+A\bar{z}|=R, \operatorname{Im}(z+A\bar{z})\ge 0\right\}$.

Evidence. Denote by

$$\gamma' R=\left\{z+A\bar{z}=\operatorname{Re}^{it}, 0\le t\le\frac{\pi}{2}\right\}$$

and

$$\gamma'' R=\left\{z+A\bar{z}=\operatorname{Re}^{it}, \frac{\pi}{2}\le t\le\pi\right\}.$$

By Jordan's inequality (i.e., $\sin t\ge\frac{2}{\pi}t, \forall t\in\left[0;\frac{\pi}{2}\right]$) and the equality $\left|dz+Ad\bar{z}\right|=\left|d(z+Ad\bar{z})\right|=Rdt$, we have

$$\left|\int_{\gamma' R} f(z)e^{i\lambda(z+A\bar{z})}(dz+Ad\bar{z})\right|\le\int_{\gamma' R}|f(z)|\left|e^{i\lambda(z+A\bar{z})}\right|\left|dz+Ad\bar{z}\right|=\int_0^{\frac{\pi}{2}}|f(z)|e^{-\lambda R\sin t}Rdt\le$$

$$\le RM(R)\int_0^{\frac{\pi}{2}}e^{-\lambda R\frac{2}{\pi}t}dt=M(R)\frac{\pi}{2\lambda}\left(1-e^{-\lambda R}\right)\underset{R\to\infty}{\to}0.$$

By Jordan's inequality, $\sin(\pi - t) \geq \frac{2}{\pi}(\pi - t)$ for all $t \in \left[\frac{\pi}{2}, \pi\right]$. Then we similarly show that

$$\left|\int_{\gamma' R} f(z)e^{i\lambda(z+A\bar{z})}(dz + Ad\bar{z})\right| \leq M(R)\frac{\pi}{2\lambda}\left(1 - e^{-\lambda R}\right) \underset{R\to\infty}{\to} 0.$$

Means

$$\left|\int_{\gamma R} f(z)e^{i\lambda(z+A\bar{z})}(dz + Ad\bar{z})\right| \leq \left|\int_{\gamma' R} f(z)e^{i\lambda(z+A\bar{z})}(dz + Ad\bar{z})\right| + \left|\int_{\gamma'' R} f(z)e^{i\lambda(z+A\bar{z})}(dz + Ad\bar{z})\right| \leq$$

$$\leq M(R)\frac{\pi}{\lambda}\left(1 - e^{-\lambda R}\right) \underset{R\to\infty}{\to} 0.$$

The lemma is proved.

Theorem 7. If $f \in O_A\left(C \setminus \{a_1, a_2 ..., a_n\}\right)$, $f|_R$ is a real function and $\lim_{R\to\infty} R \max_{|z+A\bar{z}|=R} |f(z)| = 0$, then the following equality holds

$$\int_{-\infty}^{+\infty} f(x)dx = 2\pi i \sum_{a_k \in \left\{\mathrm{Im}(z+A\bar{z}) \geq 0\right\}} \underset{z=a_k}{res_A} f(z + A\bar{z})$$

Evidence. The function $f(z + A\bar{z})$ is A-analytic in the region $D = \left\{z \in C : \left|z + A\bar{z}\right| > 0\right\}$ everywhere, except for an isolated set of singular points. By Theorem 4

$$\int_{-R}^{R} f(x)dx + \int_{\gamma R} f(z + A\bar{z})(dz + Ad\bar{z}) = \oint_{\Gamma_R} f(z)(dz + Ad\bar{z}) = 2\pi i \sum_{a_k \in \left\{\mathrm{Im}(z+A\bar{z}) \geq 0\right\}} \underset{z=a_k}{res_A} f(z + A\bar{z})$$

where

$$\Gamma_R = \left\{ z \in C : \left| z + A\bar{z} \right| = R, \operatorname{Im}(z + A\bar{z}) \geq 0 \right\} \cup \left\{ z \in C : \left| z + A\bar{z} \right| \leq R, \operatorname{Im}(z + A\bar{z}) = 0 \right\}.$$

Now we show that $\int\limits_{\gamma R} f(z + A\bar{z})(dz + Ad\bar{z}) \underset{R\to 0}{\rightarrow} 0.$

$$\left| \int\limits_{\gamma R} f(z + A\bar{z})(dz + Ad\bar{z}) \right| \leq \int\limits_{\gamma R} \left| f(z + A\bar{z}) \right| \left| dz + Ad\bar{z} \right| \leq M(R) \int\limits_0^{\pi} R dt = \pi R M(R) \underset{R\to 0}{\rightarrow} 0$$

The theorem is proved.

Consequence. If P, Q - polynomial and $\deg Q \geq \deg P + 2$ then the following equality holds

$$\int\limits_{-\infty}^{+\infty} \frac{P(x)}{Q(x)} dx = 2\pi i \sum_{a_k \in \left\{ \operatorname{Im}(z + A\bar{z}) \geq 0 \right\}} \underset{z = a_k}{res_A} \frac{P(z + A\bar{z})}{Q(z + A\bar{z})}.$$

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