MULTIDIMENSIONAL FRACTIONAL FREE ELECTRON LASER EQUATIONS WITH GENERALIZED MITTAG-LEFFLER FUNCTION

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ABSTRACT

In the present paper we consider non-homogeneous forms of fractional free electron laser (FFEL) equations with generalized Mittag-Leffler function, in pulse propagation and transverse mode cases, with Caputo fractional derivatives. We apply Adomian decomposition method to derive the closed form solution of these equations.

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1. INTRODUCTION

The unsaturated behavior of the free electron laser (FEL), when no field mode structure is taken into account, is described by the first-order integro-differential equation of Volterra type [10, 12] in the form

$$\frac{d}{d\tau}a(\tau) = -i\pi g_0 \int_0^\tau \xi e^{i\nu\xi} a(\tau-\xi)\,d\xi, 0 \le \tau \le 1, \tag{1}$$

where τ is a dimensionless time variable, g_0 is a positive constant known as the small signal gain and the real constant ν is the detuning parameter. The function $a(\tau)$ is complex-field amplitude which is assumed to be dimensionless, satisfies the condition $a(0)=1$.

The fractional form of equation (1) and its various generalizations have been studied by Boyadjiev*et al*. [5], Al-Shammery*et al*. [3,4] and Saxena and Kalla[22].

The generalization of the single mode FEL equation given by (1) to the pulse propagation case has the form [6]:

$$\frac{\partial}{\partial\tau}a(z,\tau) = -i\pi g_0 \int_0^\tau \xi e^{i\bar{\nu}\xi} a(z,\tau-\xi)\,d\xi, 0 \le \tau \le 1, \tag{2}$$

where $\Delta(0<\Delta<1)$ is the so-called slippage length, z is the longitudinal coordinate of propagation and $\bar{\nu} = \nu - i\Delta\frac{\partial}{\partial z}$. Equation (2) describes an FEL interaction which amplifies an optical pulse $a(z,\tau)$ initially given by $a(z)$.

Another generalization of the single mode FEL equation (1) to the case of transverse mode in 3-dimension has the following form [6]:

$$\left(\frac{\partial}{\partial\tau}+i\frac{S}{4}\nabla_1^2\right)g(x,y,\tau)=-i\pi g_0\int_0^{\tau}\xi e^{i\nu\xi}g(x,y,\tau-\xi)\,d\xi, \tag{3}$$

where $\nabla_1^2\equiv\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$ is the transverse Laplacian operator and S is a positive constant.

Boyadjiev and Dobner [6] studied the fractional generalizations of FEL equations given by (2) and (3)with Riemann-Liouville fractional derivative and the initial conditions were taken in the form of values of fractional derivatives at $\tau=0$, as follows:

The FFEL equation in pulse propagation case [6]:

$$D_\tau^\alpha a(z,\tau)=\lambda\int_0^{\tau}\xi e^{i\overline{\nu}\xi}a(z,\tau-\xi)\,d\xi, \qquad 0\le\tau\le 1,\lambda\in\square,\nu\in\square \tag{4}$$

with initial conditions

$$D_\tau^{\alpha-k}a(z,\tau)\big|_{\tau=0}=b_k a(z),k=1,2,...,n, \tag{5}$$

where $n-1<\alpha\le n,n\in\square,b_k$'s are given real numbers and $a(z,\tau)$ is given initially by $a(z)$ as:

$$a(z)=\frac{1}{(2\pi)^{1/4}\sigma_E^{1/2}}\exp\left(-\frac{z^2}{4\sigma_E^2}\right). \tag{6}$$

The FFEL equation in transverse mode case is given by [6]

$$\sum_{k=0}^{\infty}\binom{\alpha}{k}\left(\frac{iS}{4}\nabla_1^2\right)^k D_\tau^{\alpha-k}g(x,y,\tau)=\lambda\int_0^{\tau}\xi g(x,y,\tau-\xi)e^{i\nu\xi}d\xi, \qquad 0\le\tau\le 1,\lambda\in\square,\nu\in\square,S>0 \tag{7}$$

with initial conditions

$$D_\tau^{\alpha-k}g(x,y,\tau)\big|_{\tau=0}=\sum_{j=0}^{\infty}\binom{\alpha-k}{j}b_{j+k}\left(-\frac{iS}{4}\nabla_1^2\right)^j g(x,y),k=1,2,...,n \tag{8}$$

where $n-1<\alpha\le n,n\in\square$ and $\{b_k\}_{k=1}^{\infty}$ is a prescribed sequence of real numbers and the function $g(x,y,\tau)$ is given initially by $g(x,y)$ as:

$$g(x,y)=\frac{1}{2\pi}\exp\left(-\frac{x^2+y^2}{2}\right). \tag{9}$$

The fractional derivative D_τ^α in (4), (5), (7) and (8) are considered as Riemann-Liouville fractional derivative defined by (3).

Further generalizations of these FFEL equations have been studied by Garg [14] and Garg et al. [15] in pulse propagation case as

$${}^{C}D_\tau^\alpha a(z,\tau)=\lambda\int_0^{\tau}\xi^{\delta}\phi(b,\delta+1;i\overline{\nu}\xi)a(z,\tau-\xi)\,d\xi+\mu\tau^{\gamma}\phi(\beta,\gamma+1;i\overline{\nu}\tau)a(z),0\le\tau\le 1, \tag{10}$$

where ${}^{C}D_\tau^\alpha$ is the Caputo fractional derivative, $\mu,\lambda\in\square,\nu,b,\beta\in\square,\gamma>-1,\delta>-1,\alpha>0$, ϕ is confluent hypergeometric function [13]and initial conditions are,

$$D_\tau^k a(z,\tau)\big|_{\tau=0}=b_k a(z),k=0,1,2,...,n-1, \tag{11}$$

where $n-1<\alpha \le n, n \in \mathbb{N}, b_k$'s are given real numbers and $a(z)$ is as given by (6).

In transverse mode case as

$$\sum_{k=0}^{\infty}\binom{\alpha}{k}\left(\frac{iS}{4}\nabla_1^2\right)^k {}^{C}D_\tau^\alpha g(x,y,\tau)=\lambda\int_0^\tau t^\delta \phi(b,\delta+1;i\nu t)\, g(x,y,\tau-t)\,dt+\mu\tau^\gamma\phi(\beta,\gamma+1;i\nu\tau)\, g(x,y), 0\le\tau\le 1, \quad (12)$$

where constants $\lambda,\mu \in \mathbb{C}, \nu, b, \beta \in \mathbb{R}; \delta>-1, \gamma>-1, S>0$ and $\nabla_1^2 \equiv \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$ is transverse Laplace operator with initial conditions

$$D_\tau^k g(x,y,\tau)\big|_{\tau=0}=\sum_{j=0}^{\infty}\binom{k}{j} b_{j+k}\left(-\frac{iS}{4}\nabla_1^2\right)^j g(x,y), k=0,1,2,...,n-1, \quad (13)$$

where $n-1<\alpha\le n, n\in\mathbb{N}$ and $\{b_k\}_{k=0}^{\infty}$ is a prescribed sequence of real numbers and $g(x,y)$ is as given by (9)

In the present paper we consider generalized FFEL equation in pulse propagation and transverse mode cases involving generalized Mittag-Leffler function and solve them using the Adomian decomposition method.

The Adomian decomposition method has been introduced and developed by Adomian [1,2].Itis useful for obtaining closed form or numerical approximation for a wide class of stochastic and deterministic problems in science and engineering.This methodhas further been modified by Wazwaz [23] and Luo [17]. A considerable amount of research work has been invested recently in applying this method to a wide class of linear and nonlinear ordinary differential equations including algebraic, differential, integral, integro-differential, partial differential equations (PDEs) and systems, higher order ordinary differential equations, and others. For more details we refer [1, 2, 8, 9, 18, 23-25] and the references there in.

The paper is organized as follows. In Section 2, we give the definitions and numerical scheme used in consequent sections. In Section 3, we give solution of generalized FFEL equation in pulse propagation case with Caputo fractional derivative and initial conditions consisting of integer-order derivatives. In Section 4, we give solution of generalized FFEL equation in transverse mode case with Caputo fractional derivative.

2. PRELIMINARIES

Riemann-Liouville fractional integral of order $\alpha>0$ is defined as [18]

$$J_x^\alpha f(x)=\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t)\,dt, t>0, \quad (1)$$

$$J_x^0 f(x)=f(x). \quad (2)$$

Riemann-Liouville fractional derivative of order $\alpha>0$ is defined as the left inverse of Riemann-Liouville fractional integral[18], i.e.

$$D_t^\alpha f(t)=J_t^{-\alpha} f(t)=D^n J_t^{n-\alpha} f(t)=\frac{d^n}{dt^n}\left\{\frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-n}}d\tau\right\}, n-1<\alpha\le n, n\in\mathbb{N}. \quad (3)$$

Caputo fractional derivative of order $\alpha>0$ for a function $f(x)$ with $x\in\mathbb{R}^+$ is defined as [7]:

$$ {}^{C}D_x^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{D^n f(\xi)}{(x-\xi)^{\alpha+1-n}} d\xi, \mathrm{n}-1 < \alpha \le n, n \in \square \ . \tag{4} $$

For Riemann-Liouville fractional integral and Caputo fractional derivative, we have the following relation

$$ J_x^{\alpha}\, {}^{C}D_x^{\alpha} f(x) = f(x) - \sum_{k=0}^{n-1} D^k f(x)\Big|_{x\to 0^+} \frac{x^k}{k!}. \tag{5} $$

The Mittag-Leffler function has gained importance and popularity due to its applications in solutions of fractional-order differential, integral, integro-differential and difference equations arising in several problems of applied sciences such as physics, chemistry, biology and engineering. This function was introduced by the Swedish mathematician Mittag-Leffler[20] in terms of the following power series

$$ E_{\alpha}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\alpha n+1)}, \ x, \alpha \in \square, \ \mathrm{Re}(\alpha) > 0. \tag{6} $$

The Mittag-Leffler function (6) reduces to the exponential function when $\alpha = 1$. For $0 < \alpha < 1$ it interpolates between the pure exponential e^x and the geometric function $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n; \ (|x| < 1)$.

A generalization of (6) was studied by Wiman[26] in the form

$$ E_{\alpha,\beta}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\alpha n+\beta)}, \ x, \alpha, \beta \in \square, \ \mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0. \tag{7} $$

A further generalization of (7) was studied by Prabhakar [21] as

$$ E_{\alpha,\beta}^{\gamma}(x) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n+\beta)} \frac{x^n}{n!}, \ x, \alpha, \beta, \gamma \in \square \ , \ \mathrm{Re}(\alpha) > 0, \tag{8} $$

Adomian decomposition method for linear differential/ integro-differential equations [19, 24]:

We consider the linear differential equation written in an operator form as

$$ Lu + Ru = g, \tag{9} $$

where L is, mostly, the lower order derivative which is assumed to be invertible, R is other linear differential operator, and g is a source term.

We next apply the inverse operator L^{-1} to both sides of equation (9) and use the given conditions to obtain

$$ u = f - L^{-1}(Ru), \tag{10} $$

where the function f represents the terms that arise due toapplication of L^{-1} to the source term g and using the givenconditions that are assumed to be prescribed.Further we decompose the unknown function u into a sum of an infinite number of components given by the decomposition series

$$ u = \sum_{m=0}^{\infty} u_m, \tag{11} $$

where the components $u_0, u_1, u_2, ...$ are usually recurrently determined. Substituting (11) into both sides of (10) leads to

$$ \sum_{m=0}^{\infty} u_m = f - L^{-1}\left(R\left(\sum_{m=0}^{\infty} u_m \right) \right). \tag{12} $$

This can be written as

$$u_0 + u_1 + u_2 + u_3 + ... = f - L^{-1}\left(R\left(u_0 + u_1 + u_2 + ...\right)\right). \tag{13}$$

Adomian method uses the formal recursive relation as

$$u_0 = f,$$
$$u_{l+1} = -L^{-1}\left(R\left(u_l\right)\right), l = 0,1,2,.... \tag{14}$$

3. SOLUTION OF GENERALIZED FRACTIONAL FREE ELECTRON LASER EQUATION IN PULSE PROPAGATION CASE

Theorem1. For $\rho, \mu, \lambda \in \mathbb{C}, \nu, b, \beta \in \mathbb{R}, \gamma > -1, \delta > -1, \alpha > 0$, consider the generalized FFEL equation

$$ {}^{C}D_{\tau}^{\alpha} a(z,\tau) = \lambda \int_0^{\tau} \xi^{\delta-1} E_{\rho,\delta}^{b}\left(i\bar{\nu}\xi^{\rho}\right) a(z,\tau-\xi)\, d\xi + \mu\tau^{\gamma-1} E_{\rho,\gamma}^{\beta}\left(i\bar{\nu}\tau^{\rho}\right) a(z), 0 \le \tau \le 1, \tag{1}$$

where $E_{\rho,\delta}^{b}(x)$ is the generalized Mittag-Leffler function given by (8)and initial conditions are,

$$D_{\tau}^{k} a(z,\tau)\big|_{\tau=0} = b_k a(z), k = 0,1,2,...,n-1, \tag{2}$$

where $n-1 < \alpha \le n, n \in \mathbb{N}, b_k$'s are given real numbers and

$$a(z) = \frac{1}{(2\pi)^{1/4}\sigma_E^{1/2}} \exp\left(-\frac{z^2}{4\sigma_E^2}\right). \tag{3}$$

The closed form solution to this problem is given by

$$\begin{aligned} a(z,\tau) &= \sum_{k=0}^{n-1} \frac{b_k \tau^k}{k!} a(z) + \sum_{k=0}^{n-1} b_k \tau^k \sum_{m=1}^{\infty} \lambda^m \tau^{m(\alpha+\delta)} \sum_{r=0}^{\infty} \frac{(mb)_r}{r!} \left(-\frac{\tau^{\rho}\Delta}{2\sigma_E}\right)^r E_{\rho,m(\alpha+\delta)+k+1+r}^{mb+r}\left(i\nu\tau^{\rho}\right) H_r\left(\frac{z}{2\sigma_E}\right) a(z) \\ &+ \mu \sum_{m=0}^{\infty} \lambda^m \tau^{m(\alpha+\delta)+\gamma+\alpha-1} \sum_{r=0}^{\infty} \frac{(\beta+mb)_r}{r!} \left(-\frac{\tau^{\rho}\Delta}{2\sigma_E}\right)^r E_{\rho,m(\alpha+\delta)+\alpha+\gamma+r}^{\beta+mb+r}\left(i\nu\tau^{\rho}\right) H_r\left(\frac{z}{2\sigma_E}\right) a(z), \end{aligned} \tag{4}$$

where $H_r(*)$ are Hermite polynomials [13].

Proof. We substitute $s = \tau - \xi$ in (1)so that it transforms to

$$ {}^{C}D_{\tau}^{\alpha} a(z,\tau) = \mu\tau^{\gamma-1} E_{\rho,\gamma}^{\beta}\left(i\bar{\nu}\tau^{\rho}\right) a(z) + \lambda \int_0^{\tau} (\tau - s)^{\delta-1} E_{\rho,\delta}^{b}\left(i\bar{\nu}(\tau-s)^{\rho}\right) a(z,s)\, ds, \tag{5}$$

Applying J_{τ}^{α} on both sides of equation (5) and using relation(5)with initial conditions(2), we get

$$a(z,\tau) = \sum_{k=0}^{n-1} \frac{b_k \tau^k}{k!} a(z) + J_{\tau}^{\alpha}\left[\mu\tau^{\gamma-1} E_{\rho,\gamma}^{\beta}\left(i\bar{\nu}\tau^{\rho}\right) a(z) + \lambda \int_0^{\tau} (\tau-s)^{\delta-1} E_{\rho,\delta}^{b}\left(i\bar{\nu}(\tau-s)^{\rho}\right) a(z,s)\, ds\right]. \tag{6}$$

Further we decompose the unknown function $a(z,\tau)$ into a sum of an infinite number of components given by the decomposition series

$$a(z,\tau) = \sum_{m=0}^{\infty} a_m(z,\tau). \tag{7}$$

Using Adomian decomposition method,these components can recursively be obtained by

$$a_0(z,\tau)=\sum_{k=0}^{n-1}\frac{b_k\tau^k}{k!}a(z)+J_\tau^\alpha\left[\mu\tau^{\gamma-1}E_{\rho,\gamma}^{\beta}\left(i\bar{\nu}\tau^\rho\right)a(z)\right],$$
$$a_{l+1}(z,\tau)=J_\tau^\alpha\left[\int_0^\tau\lambda(\tau-s)^{\delta-1}E_{\rho,\delta}^{b}\left(i\bar{\nu}(\tau-s)^\rho\right)a_l(z,s)ds\right],l=0,1,2,3,.... \quad (8)$$

Making use of Dirichlet's formula given as

$$\int_0^t\int_0^\tau u(\tau,s)dsd\tau=\int_0^t\int_s^t u(\tau,s)d\tau ds \quad (9)$$

and integral summation formula for the Mittag-Leffler type function $E_{b,c}^{a}(t)$ [16]

$$\int_0^t\xi^{c-1}E_{b,c}^{a}\left(k\xi^r\right)(t-\xi)^{c'-1}E_{b,c'}^{a'}\left(k(t-\xi)^r\right)d\xi=t^{c+c'-1}E_{b,c+c'}^{a+a'}\left(kt^r\right), \quad (10)$$

where $a,b,c,a',c'\in\mathbb{C}$, $\mathrm{Re}(b),\mathrm{Re}(c),\mathrm{Re}(c')>0$.

in recursive formula (8), we obtain these components as

$$a_m(z,\tau)=\lambda^m\left[\sum_{k=0}^{n-1}b_k\tau^{m(\alpha+\delta)+k}E_{\rho,m(a+\delta)+k+1}^{mb}\left(i\bar{\nu}\tau^\rho\right)a(z)+\mu\tau^{m(\alpha+\delta)+\gamma+\alpha-1}E_{\rho,m(a+\delta)+\gamma+\alpha}^{\beta+mb}\left(i\bar{\nu}\tau^\rho\right)a(z)\right],m=0,1,2,3,...$$

(11)

Now to simplify the function $E_{\beta,\gamma}^{\alpha}$ occurring in (11), we write $\bar{\nu}=\nu-i\Delta\frac{\partial}{\partial z}$, use series manipulation and obtain after a little simplification the following results:

$$E_{\rho,\alpha+\gamma}^{\beta}\left(i\bar{\nu}\tau^\rho\right)a(z)=\sum_{r=0}^{\infty}\frac{(\beta)_r}{r!}\left(-\frac{\tau^\rho\Delta}{2\sigma_E}\right)^r E_{\rho,\alpha+\gamma+r}^{\beta+r}\left(i\nu\tau^\rho\right)H_r\left(\frac{z}{2\sigma_E}\right)a(z),$$

$$E_{\rho,m(\alpha+\delta)+k+1}^{mb}\left(i\bar{\nu}\tau^\rho\right)a(z)=\sum_{r=0}^{\infty}\frac{(mb)_r}{r!}\left(-\frac{\tau^\rho\Delta}{2\sigma_E}\right)^r E_{\rho,m(\alpha+\delta)+k+r+1}^{mb+r}\left(i\nu\tau^\rho\right)H_r\left(\frac{z}{2\sigma_E}\right)a(z)$$

and

$$E_{\rho,m(\alpha+\delta)+\gamma+\alpha}^{\beta+mb}\left(i\bar{\nu}\tau^\rho\right)a(z)=\sum_{r=0}^{\infty}\frac{(\beta+mb)_r}{r!}\left(-\frac{\tau^\rho\Delta}{2\sigma_E}\right)^r E_{\rho,m(\alpha+\delta)+\gamma+\alpha+r}^{\beta+mb+r}\left(i\nu\tau^\rho\right)H_r\left(\frac{z}{2\sigma_E}\right)a(z). \quad (12)$$

Substitung(12) in (11) and using the result in (7), we obtain the required result as given in equation(4).

Remarks:

We observe that on specializing various parameters in Theorem 1 as $\rho=1$, replacing δ by $\delta-1$, γ by $\gamma-1$, the FFEL (1)-(3) and its solution (4) reduce to the one solved by Garg [14] and Garg et al. [15] and on specializing the parameters further, the sub cases studied by Garg [14] and Garg et al. [15] can be obtained.

4. THE GENERALIZED FFEL EQUATION IN TRANSVERSE MODE CASE

Theorem2.*Consider the generalized FFEL equation*

$$\sum_{k=0}^{\infty}\binom{\alpha}{k}\left(\frac{iS}{4}\nabla_1^2\right)^k {}^{C}D_\tau^\alpha g\left(x,y,\tau\right)=\lambda\int_0^\tau t^{\delta-1}E_{\rho,\delta}^{b}\left(i\nu t^\rho\right)g\left(x,y,\tau-t\right)dt+\mu\tau^{\gamma-1}E_{\rho,\gamma}^{\beta}\left(i\nu\tau^\rho\right)g\left(x,y\right),0\le\tau\le1,$$

(1)

where constants $\lambda,\mu,\rho\in\Box,\nu,b,\beta\in\Box;\delta>-1,\gamma>-1,S>0$ *and* $\nabla_1^2\equiv\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$ *is transverse Laplace operator with initial conditions*

$$D_\tau^k g\left(x,y,\tau\right)\Big|_{\tau=0}=\sum_{j=0}^{\infty}\binom{k}{j}b_{j+k}\left(-\frac{iS}{4}\nabla_1^2\right)^j g\left(x,y\right),k=0,1,2,...,n-1, \tag{2}$$

where $n-1<\alpha\le n,n\in\Box$ *and* $\{b_k\}_{k=0}^{\infty}$ *is a prescribed sequence of real numbers and*

$$g\left(x,y\right)=\frac{1}{2\pi}\exp\left(-\frac{x^2+y^2}{2}\right). \tag{3}$$

The closed form solution to this problem is given by

$$\begin{aligned}g\left(x,y,\tau\right)=\frac{1}{2\pi}e^{-\frac{\left(x^2+y^2\right)}{2}-\frac{iS\tau}{4}\nabla_1^2}\Bigg[&\sum_{k=0}^{n-1}\frac{b_k\tau^k}{k!}+\sum_{k=0}^{n-1}\frac{b_k\tau^k}{k!}\sum_{m=1}^{\infty}\lambda^m\tau^{m(\alpha+\delta)}\sum_{r=0}^{\infty}\left(mb\right)_r\left(-\frac{iS\tau^\rho}{2}\right)^r E_{\rho,m(\alpha+\delta)+k+r+1}^{mb+r}\left(i\nu\tau^\rho\right)L_r\left(\frac{x^2+y^2}{2}\right)\\&+\mu\sum_{m=0}^{\infty}\lambda^m\tau^{m(\alpha+\delta)+\gamma-1+\alpha}\sum_{r=0}^{\infty}\left(\beta+mb\right)_r\left(-\frac{iS\tau^\rho}{2}\right)^r E_{\rho,m(\alpha+\delta)+\gamma+\alpha+r}^{\beta+mb+r}\left(i\nu\tau^\rho\right)L_r\left(\frac{x^2+y^2}{2}\right)\Bigg],\end{aligned}$$

(4)

where $L_r(*)$ are Laguerre polynomials [13].

ProofWe substitute $t=\tau-\xi$ in (1)and introduce an unknown function $G\left(x,y,\tau\right)$ defined as follows

$$g\left(x,y,\tau\right)=e^{-\frac{iS\tau}{4}\nabla_1^2}G\left(x,y,\tau\right). \tag{5}$$

Equation(1) reduces to the form

$${}^{C}D_\tau^\alpha G\left(x,y,\tau\right)=\lambda\int_0^\tau\left(\tau-\xi\right)^{\delta-1}E_{\rho,\delta}^{b}\left(i\bar{\nu}\left(\tau-\xi\right)^\rho\right)G\left(x,y,\xi\right)d\xi+\mu\tau^{\gamma-1}E_{\rho,\gamma}^{\beta}\left(i\bar{\nu}\tau^\rho\right)g\left(x,y\right),0\le\tau\le1, \tag{6}$$

where $\bar{\nu}=\nu+\frac{S}{4}\nabla_1^2$, with initial conditions

$$D_\tau^k G\left(x,y,\tau\right)\Big|_{\tau=0}=b_k,k=0,1,2,...,n-1. \tag{7}$$

Applying J_τ^α on both sides of equation (6) and using equation (5) with initial conditions(7), we get

$$G\left(x,y,\tau\right)=\sum_{k=0}^{n-1}\frac{b_k\tau^k}{k!}g\left(x,y\right)+J_\tau^\alpha\left[\mu\tau^{\gamma-1}E_{\rho,\gamma}^{\beta}\left(i\bar{\nu}\tau^\rho\right)g\left(x,y\right)+\int_0^\tau\lambda\left(\tau-\xi\right)^{\delta-1}E_{\rho,\delta}^{b}\left(i\bar{\nu}\left(\tau-\xi\right)^\rho\right)G\left(x,y,\xi\right)d\xi\right].$$

(8)

Further we decompose the unknown function $G\left(x,y,\tau\right)$ into a sum of an infinite number of components given by the decomposition series

$$G(x,y,\tau)=\sum_{m=0}^{\infty}G_m(x,y,\tau). \tag{9}$$

Using Adomian decomposition method, these components can recursively be obtained by

$$\begin{aligned}
G_0(x,y,\tau)&=\sum_{k=0}^{n-1}\frac{b_k\tau^k}{k!}g(x,y)+J_\tau^\alpha\left[\mu\tau^{\gamma-1}E_{\rho,\gamma}^{\beta}\left(i\bar{\nu}\tau^\rho\right)g(x,y)\right],\\
G_{l+1}(x,y,\tau)&=J_\tau^\alpha\left[\int_0^\tau\lambda(\tau-\xi)^{\delta-1}E_{\rho,\delta}^{b}\left(i\bar{\nu}(\tau-\xi)^\rho\right)G_l(x,y,\xi)\,d\xi\right],l=0,1,2,3,....
\end{aligned} \tag{10}$$

Making use of Dirichlet's formula(9)and integral summation formula (10)in recursive formula(10), we obtain these components as

$$\begin{aligned}
G_0(x,y,\tau)&=\sum_{k=0}^{n-1}\frac{b_k\tau^k}{k!}g(x,y)+\mu\tau^{\alpha+\gamma-1}E_{\rho,\alpha+\gamma}^{\beta}\left(i\bar{\nu}\tau^\rho\right)g(x,y),\\
G_m(x,y,\tau)&=\lambda^m\left[\sum_{k=0}^{n-1}b_k\tau^{m(\alpha+\delta)+k}E_{\rho,m(\alpha+\delta)+k+1}^{mb}\left(i\bar{\nu}\tau^\rho\right)g(x,y)+\mu\tau^{m(\alpha+\delta)+\gamma-1+\alpha}E_{\rho,m(\alpha+\delta)+\gamma+\alpha}^{mb}\left(i\bar{\nu}\tau^\rho\right)g(x,y)\right],m=1,2,3,...
\end{aligned} \tag{11}$$

On simplification, equation (11) reduces to

$$\begin{aligned}
G_0(x,y,\tau)&=\sum_{k=0}^{n-1}\frac{b_k\tau^k}{k!}g(x,y)+\mu\tau^{\alpha+\gamma-1}\sum_{r=0}^{\infty}(\beta)_r\left(-\frac{iS\tau^\rho}{2}\right)^r E_{\rho,\alpha+\gamma+r}^{\beta+r}\left(i\nu\tau^\rho\right)L_r\left(\frac{x^2+y^2}{2}\right)g(x,y),\\
G_m(x,y,\tau)&=\lambda^m\left[\sum_{k=0}^{n-1}b_k\tau^{m(\alpha+\delta)+k}\sum_{r=0}^{\infty}(mb)_r\left(-\frac{iS\tau^\rho}{2}\right)^r E_{\rho,m(\alpha+\delta)+k+1}^{mb+r}\left(i\nu\tau^\rho\right)L_r\left(\frac{x^2+y^2}{2}\right)g(x,y)\right.\\
&\left.+\mu\tau^{m(\alpha+\delta)+\gamma-1+\alpha}\sum_{r=0}^{\infty}(\beta+mb)_r\left(-\frac{iS\tau^\rho}{2}\right)^r E_{\rho,m(\alpha+\delta)+\gamma+\alpha+r}^{\beta+mb+r}\left(i\nu\tau^\rho\right)g(x,y)\right],m=1,2,3,...
\end{aligned} \tag{12}$$

Substituting (12) in (9) and using the result in(5), we obtain the required result as given in equation(4).

Remarks:

We observe that on specializing various parameters in Theorem 2 as $\rho=1$, replacing δ by $\delta-1$, γ by $\gamma-1$, the FFEL (1)-(3) and its solution (4) reduce to the one solved by Garg [14] and Garg et al. [15] and on specializing the parameters further, the sub cases studied by Garg [14] and Garg et al. [15] can be obtained.

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