STUDY OF DIFFERENTIABLE MANIFOLDS AND THEIR APPLICATIONS BASED ON THEOREMS

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Abstract

A differentiable manifold is a topological manifold with a globally defined differential structure. On a manifold that is sufficiently smooth, various kinds of jet bundles can also be considered. The tangent bundle of a manifold is the collection of curves in the manifold that is equivalent to the relation of first-order contact. Therefore, by analogy the *k*-th order tangent bundle is the collection of curves of the relation of *k*-th order contact. Likewise, the cotangent bundle is the bundle of one-jets of functions on the manifold: the *k*-jet bundle is the bundle of their *k*-jets. These and other examples of the general idea of jet bundles play a significant role in the study of differential operators on manifolds. It is worth noting that every topological manifold in *n* dimensions has a unique differential structure (up to diffeomorphism). Thus, the concepts of topological and differentiable manifold are only distinct in higher dimensions. It is known that in each higher dimension, there are some topological manifolds with no smooth structure, and some with multiple non-diffeomorphic structures. The Gauss-Bonnet theorem surprisingly implies that the Euler number does not depend on the vector field X, and the integral of the curvature with respect to the area element does not depend on the metric. Therefore, each of these concepts are inherent to the structure of the manifold. Another interesting consequence of this theorem is that two compact manifolds without boundary M2 and N2 are diffeomorphic, in other words, there exists a smooth function with a smooth inverse between M2 and N2, if and only if X(M2) = X(N2). In other words, if we can smoothly identify one manifold with another, then their Euler numbers are the same. This proof is primarily due to Shiing Shen Chern, who in fact used this technique to prove a generalized version of the Gauss-Bonnet theorem that holds for higher dimensional manifolds.

Key Words: Differentiable Manifolds, Applications, Theorems, Elements, homeomorphisms.

Introduction:

Formally, the word "differentiable" is somewhat ambiguous, as it is taken to mean different things by different authors; sometimes it means the existence of first derivatives, sometimes the existence of continuous first derivatives, and sometimes the existence of infinitely many derivatives [1]. The following gives a formal definition of various (nonambiguous) meanings of "differentiable atlas". Generally, "differentiable" will be used as a catch-all term including all of these possibilities, provided $k \geq 1$. Differentiability means different things in different contexts including: continuously differentiable, k times differentiable, smooth, and holomorphic. Furthermore, the ability to induce such a differential structure on an abstract space allows one to extend the definition of differentiability to spaces without global coordinate systems. A differential structure allows one to define the globally differentiable tangent space, differentiable functions, and differentiable tensor and vector fields[2]. Differentiable manifolds are very important in physics. Special kinds of differentiable manifolds form the basis for physical theories such as classical mechanics, general relativity, and Yang–Mills theory. It is possible to develop a calculus for differentiable manifolds. This leads to such mathematical machinery as the exterior calculus. The study of calculus on differentiable manifolds is known as differential geometry[3].

We first introduce the concept of a manifold, which leads to a discussion of differential forms, the exterior derivative and pull-back map. We then discuss integration of forms in Rn in order to state and prove Stokes' Theorem in Rn. A few applications of Stokes' Theorem are also stated and proved, such as Brouwer's fixed point theorem. In order to discuss Chern's proof of the Gauss-Bonnet Theorem in R3, we slightly shift gears to discuss geometry in R3. We introduce the concept of a Riemannian Manifold and develop Elie Cartan's Structure Equations in Rn to define Gaussian Curvature in R3. The Poincarre-Hopf Index Theorem is first stated and proved, and the concept of the Euler number is introduced in order to end with a proof of the Gauss-Bonnet Theorem in R3. A few important implications of the theorem are then mentioned[4].

Most of the definitions, along with proofs of the propositions and theorems have been adapted from Do Carmo's Differential Forms and Applications [5], along with Pressley's Elementary Differential Geometry which are as follows:

Definition:1 A topology on a set X is a collection T of subsets of X such that ϕ and X are in T.

The union of an arbitrary collection of elements of T is in T.

The intersection of a finite collection of elements of T is in T,.

Definition 2.: A basis for a topology [6] on a set X is a collection B of subsets of X such that For each $x \in X$ there exists a $B \in \mathrm{B}$ containing x.

If $B_1, B_2 \in \mathrm{B}$ and $x \in B_1 \cap B_2$, then there exists a $B_3 \subset B_1 \cap B_2$ such that.

The basis B generates a topology T by defining a set $U \subset X$ to be open if for each $x \in U$ there exists a basis element $B \in \mathrm{B}$ with $x \in B \subset 0$.

Definition 3: Let X and Y be topological spaces. The product topology on the product set $X \times Y$ is generated by the basis elements $U \times V$, for all open sets $U \in X$ and $V \in Y$,.

Topology developed from the desire to generalize the notion of continuity of mappings of Euclidean spaces [7]. That generalization is phrased as follows:

Definition 4: Let X and Y be topological spaces. A mapping $f : X \to Y$ is continuous if for each open set $U \subset Y$, the set $f^{-1}(U)$ is open in X,.

Definition 5: Let X and Y be topological spaces. A mapping $f : X \to Y$ is a homeomorphism if it is bijective and both f and f^{-1} are continuous. In this case X and Y are said to be homeomorphic [8].

When X and Y are homeomorphic, there is a bijective correspondence between both the points and the open sets of X and Y. Therefore, as topological spaces, X and Y are indistinguishable. This means that any property or theorem [9] that holds for the space X that is based only on the topology of X also holds for Y,.

Definition 6: A topological space X is said to be Hausdorff if for any two distinct points $x, y \in X$ there exist disjoint open sets U and V with $x \in U$ and $y \in V$,.

Definition7: A separation of a topological space X is a pair of disjoint open sets U, V such that $X = U \cup V$. If no separation of X exists, it is said to be connected [10].

Topological Manifolds: A manifold is a topological space that is locally equivalent to Euclidean space [11].

Definition 8: A manifold is a Hausdorff space *M* with a countable basis such that for each point $p \in M$ there is a neighborhood U of p that is homeomorphic to R^n for some integer n. If the integer n is the same for every point in *M*, then M is called as n-dimensional manifold.

Differentiable Structures of Manifolds: Differentiation of mappings in Euclidean space is defined as a local property. Although a manifold is locally homeomorphic to Euclidean space, more structure is required to make differentiation possible. Any function on Euclidean space $f : R^n \to R$ is smooth or C^∞ if all of its partial derivatives exist.

A mapping of Euclidean spaces $f : R^m \to R^n$ can be thought of real-valued functions on $R^m, f = (f^1, \ldots\ldots, f^n)$, and f is smooth if each fi is smooth.

Given two neighborhoods U, V in a manifold *M*, two homeomorphisms $x : U \to R^n$ and $y : V \to R^n$ are said to be C^∞-related if the mappings $x \circ y^{-1} : y(U \cap V) \to x(U \cap V)$ and $y \circ x^{-1} : x(U \cap V) \to y(U \cap V)$ are C^∞ .

Smooth Functions and Mapping: The pair (x, U) is called a chart or coordinate system, and can be thought of as assigning a set of coordinates to points in the neighborhood U , figure 1. A collection of charts whose domains cover M is called an atlas.

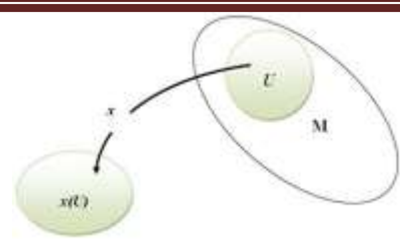

Figure 1. A local coordinate system (x, U) on a manifold M.

Definition 9: An atlas A on a manifold M is said to be maximal if for any compatible atlas A' on M any coordinate chart $(x,U) \in A'$ is also a member of A.

Definition10: A smooth structure on a manifold M is a maximal atlas A on M. The manifold M along with such an atlas is termed a smooth manifold [12].

Consider a function $f : M \to R$ on the smooth manifold M. This function is said to be a smooth function if for every coordinate chart (x,U) on M the function $f \circ x^{-1} : U \to R$ is smooth. More generally, a mapping $f : M \to N$ of smooth manifolds is said to be a smooth mapping if for each coordinate chart (x,U) on M and each coordinate chart (y,V) on N the mapping $y \circ f \circ x^{-1} : x(U) \to y(V)$ is a smooth mapping.

Definition 11: Given two smooth manifolds M, N, a bijective mapping $f : M \to N$ is called a diffeomorphism if both f and f^{-1} are smooth mappings.

There are various ways to define the derivative of a function on a differentiable manifold, the most fundamental of being the directional derivative [13]. The definition of the directional derivative is complicated by the fact that a manifold will lack a suitable affine structure with which to define vector. The directional derivative therefore looks at curves in the manifold instead of vectors [14].

Directional differentiation: Given a real valued function *f* on an *m* dimensional differentiable manifold M, the directional derivative of *f* at a point *p* in M is defined as follows. Suppose that *g* (*t*) is a curve in M with $g(0)=p$, which is differentiable in the sense that its composition with any chart is a differential curve in R^n. Then the directional derivative of *f* at *p* along *g* is $df_{(g(t))}\big|_{t=0}$. The directional derivative only depends on the tangent vector of the curve at the point consindered, p for this case. Thus a definition of directional differentiation adapted to the cases of differentiable manifolds ultimately captures the intuitive features of directional differentiation in an affine space [15].

Tangent Spaces and Derivatives:

Definition: Suppose that M is a smooth m-dimensional manifold of some Euclidean space R^n. Let $f:U\rightarrow M$ be a local parameterization around some point $x\in M$ with $\phi(0)=x$. The tangent space T_xM is the image of the map $d\phi_0:R^m\rightarrow R^n$ where T_xM is an m-dimensional subspace of R^n. The vectors in this space are called tangent vectors.

Given a smooth map of manifolds M,N, $f:M\rightarrow N$, and a local parameterization $\phi:U\rightarrow M, \phi(0)=x\in M$ and $\phi:V\rightarrow N, \psi(0)=f(x)\in N$.

Let *h* be the map $h=\psi^{-1}\circ f\circ\phi:U\rightarrow V$, the we define the differential of

f at *x* by $df_x:T_xM\rightarrow T_{f(x)}N$

$df_x=d\psi_0\circ dh_0\circ d\phi_0^{-1}$

The collection of tangent spaces at all points can in turn be made into a manifold, the tangent bundle, whose dimension is 2*n*. The tangent bundle is where tangent vectors lie, and is itself a differentiable manifold [16].

Immmersion, Submersions and Embeddings: The maps $df_x:T_xM\rightarrow T_{f(x)}N$ for all points $x\in M$ assemble to a map of tangent bundles,

Definition*:* A map f is called a submersion or we say that f is submersive if the linear map $df_x:T_xM\rightarrow T_{f(x)}N$ is an epimorphism (i.e., surjective) at each point *x*. It is called an immersion (or an immersive map) if the linear map $df_x : T_xM \to T_{f(x)}N$ is a monomorphism (i. e., injective) at each point. A smooth map $f : M \to N$ that is both injective and immersive is called embedding.

Different Theorems and Their Applications:

(i) Stokes' Theorem: It is about manifolds with boundaries, which is larger than the class of manifolds [17]. For example, a cylinder of radius r and length d oriented in the z - axis is not a manifold, because, roughly speaking, the edges of the cylinder do not locally `look' like Rn. However, the area around the edges does look like the half plane [18].

H2 = {(x1, x2) | x1 ≤ 0}.

Applications: Stokes' Theorem appears in many forms. It is, in fact, the generalized form of the fundamental theorem of calculus [19]. This can be seen in the first case of the proof, when we compute ∫M dω. This theorem is based on the Gauss-Bonnet theorem. Stokes' Theorem, as we can see, can be used to prove some important theorems. Now, in order to discuss the Gauss-Bonnet theorem in R3, we must first discuss important concepts related to geometry in R3 [20].

Therefore, in R^3, the structure equations are as follows:

dω1 = ω12 ^ ω2

dω2 = ω21 ^ ω1

dω3 = ω13 ^ ω32

dω12 = ω13 ^ ω23

dω13 = ω12 ^ ω21

dω23 = ω21 ^ ω13

The Gauss-Bonnet Theorem: We will only prove the Gauss-Bonnet theorem for two-dimensional manifolds immersed in R3. First, we prove an equivalent theorem, known as the Poincare' Hopf index theorem, and show its equivalence to Gauss Bonnet [21]. Both theorems relate topological properties of the manifold to its geometric properties [22]. With regards to topology, we would like to define the index of a vector field at a point on a manifold [23, 24].

Consider a differential vector field X defined on M. We define p ϵ M as a singular point if X (p) = 0. Further, p is isolated if there exists a neighborhood Vp c M containing p that contains no other singular point. The number of such isolated points is finite, since M is compact. We also choose V to be homeomorphic to a disk in R2, because integration is easier. We will now develop a topological property corresponding to an isolated point [25, 26].

Conclusion:

The concept of manifolds is central to many parts of geometry and modern mathematical physics because it allows more complicated structures to be expressed and understood in terms of the relatively well-understood properties of simpler spaces. Intuitively, a manifold is a generalization of curves and surfaces to higher dimensions. It is locally Euclidean in that every point has a neighborhood, called a chart, homeomorphic to an open subset. The coordinates on a chart allow one to carry out computations as though in a Euclidean space, such as differentiability, point-derivations, tangent spaces, and differential forms, carry over to a manifold. For most applications, a special kind of topological manifold, a differentiable manifold, is used. If the local charts on a manifold are compatible in a certain sense, one can define directions, tangent spaces, and differentiable functions on that manifold. In particular it is possible to use calculus on a differentiable manifold. Each point of a differentiable manifold has a tangent space. This is a Euclidean space consisting of the tangent vectors of the curves through the point. Two important classes of differentiable manifolds are smooth and analytic manifolds. For smooth manifolds the transition maps are smooth, that is infinitely differentiable. Analytic manifolds are smooth manifolds with the additional condition that the transition maps are analytic. In other words, a differentiable (or, smooth) manifold is a topological manifold with a globally defined differentiable (or, smooth) structure. A topological manifold can be given a differentiable structure locally by using the homeomorphisms in the atlas of the topological space. The global differentiable structure is induced when it can be shown that the natural compositions of the homeomorphisms between the corresponding open Euclidean spaces are differentiable on overlaps of charts in the atlas. Therefore, the coordinates defined by the homeomorphisms are differentiable with respect to each other when treated as real valued functions with respect to the variables defined by other coordinate systems whenever charts overlap. This idea is often presented formally using transition maps. This allows one to extend the meaning of differentiability to spaces without global coordinate systems. Specifically, a differentiable structure allows one to define a global differentiable tangent space, and consequently, differentiable functions, and differentiable tensor–fields. Differentiable manifolds are very important in physics. Special kinds of differentiable manifolds form the arena for physical theories such as classical mechanics, general relativity and Yang–Mills gauge theory. It is possible to develop calculus on differentiable manifolds, leading to such mathematical machinery as the exterior calculus. Historically, the development of differentiable manifolds is usually credited to C. F. Gauss and his student B. Riemann. The work of physicists J. C. Maxwell and A. Einstein lead to the development of the theory transformations between coordinate systems which preserved the essential geometric properties. Eventually these ideas were generalized by H. Weyl who essentially considered the coordinate functions in terms of other coordinates and to assume differentiability for the coordinate function.

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