I-Groups and some basic properties

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Abstract

Here we generalize abstract group structure by introducing an idempotent unary operation, called I-group, and study some of its basic properties.

1. Introduction: Group theory has broad applications in various fields of sciences. Also, various types of extensions and generalizations have been done on it. The motivation forthe introduction of I-groups comes from these observations. To introduce e concept of I-groups and some basic results, we have gone through [1-6].

2. Main results

Here we shall introduce I-group and study some of its basic properties.

Definition 2.1 Let *G* be a nonempty set on which 'o' is a binary operation and *f* is an idempotent unary operation, *i.e.*,$f: G \to G,\ f^2 = f$. The algebraic structure(G, o, f) is said to be an I-group if

(i)(G, o) is a semi-group.

(ii) there exists $e \in G$ such that $aoe = eoa = f(a), \forall a \in G$.

(iii) for every $a \in G$, there exists $b \in G$ such that $aob = boa = f(e)$.

Remarks 2.1 In Definition 1.1, such an element *e* is called an I-identity in *G*, and *b* is called an I-inverse of *a* in *G*. Clearly every group (G, o) is an I-group (G, o, I_G).

Example 2.1 Consider the set *G* of all nonzero real numbers. Let $f: G \to G$ is given by $f(x) = |\ x\ |$,$\forall x \in G$. Then $f^2 = f$ so that *f* is an idempotent unary operation on *G*. Define a binary operation 'o' on *G* by $xoy =|\ xy\ |, \forall x, y \in G$. Then (G, o, f) is an I-group with I-identities 1 and -1.

Example 2.2 Let *n* be a given positive integer and $f: \mathbb{Z} \to \mathbb{Z}$ is given by $f(x) = r_x =$ the remainder after dividing *x* by *n*. Define a binary operation $*$ on $\mathbb{Z}$ by $x * y = f(x + y), \forall\ x, y \in \mathbb{Z}$. Then $(\mathbb{Z}, *, f)$ is an I-group with I-identities $0, \pm n, \pm 2n, \pm 3n$, etc. and I-inverses $-x, -x \pm n, -x \pm 2n, -x \pm 3n$, etc. of an element *x*.

Example 2.3 Let *G* be a nonempty set and $f: G \to G$ be a constant mapping given by $f(x) = c, \forall x \in G$, where *c* is some fixed element of *G*. Define a binary operation $*$ on *G* by $x * y = c, \forall x, y \in G$.
Then $(G, *, f)$ is an I-group.

Example 2.4 Let $f: \mathbb{R} \to \mathbb{R}$ be given by $f(x) = [x] =$ the largest integer $\leq x, \forall x \in \mathbb{R}$. Define a binary operation $*$ on $\mathbb{R}$ by $a * b = f(a + b), \forall\ a, b \in \mathbb{R}$. Then $(\mathbb{R}, *, f)$ is an I-group.

Example 2.5 Let (G, o) be a group and $f: G \to G$ be an idempotent group homomorphism.
Then $(G, *, f)$ is an I-group, where $a * b = f(ab), \forall\ a, b \in G$.

Definition 2.2 Two elements *a*,*b* in an I-group (G, o, f) are said to be I-unique or I-equal if $f(a) = f(b)$; otherwise, they are called I-distinct elements.

Theorem 2.1 In an I-group (G, o, f), if e be an I-identity, then
(i) $e^n = f(e)$, for all positive integers $n \geq 2$.
(ii) $af(e) = f(e)a = f(e)f(a) = f(a)f(e) - f(a)e = ef(a) = f(a), \forall a \in G$.
(iii) $f(ab) = af(b) = f(a)b = f(a)f(b), \forall a, b \in G$.
(iv) e is I-unique.
(v) I-inverse of every element is I-unique.
Proof: (i), (ii), (iv) and (v) are trivial.
(iii) $f(ab) = (ab)e = a(be) = af(b)$.
Again, $f(ab) = e(ab) = (ea)b = f(a)b$.
Also, $f(ab) = f^2(ab) = ef(ab) = e\big((ab)e\big) = (ea)(be) = f(a)f(b)$.
Hence the result.

Theorem 2.2 In an I-group (G, o, f), for any $a, b, c \in G$,
(i) $ab = ac \implies f(b) = f(c)$ (called left I-cancellation law)
(ii) $ba = ca \implies f(b) = f(c)$ (called right I-cancellation law)
Proof: (i) Let a' be an I-inverse of a.
Now $ab = ac \implies (a'a)b = (a'a)c \implies f(e)b = f(e)c$ (e being I-identity in G)
$\implies f(b) = f(c)$.

(ii) Similar to (i).

Theorem 2.3 Let (G, o, f) be an I-group and ρ be any congruence equivalence relation on G such that $\rho \subseteq \rho_f = \{(x, y) \in G \times G : f(x) = f(y)\}$. Then $\left(\frac{G}{\rho}, *, h\right)$ is an I-group under the binary operation $*$ on G/ρ, given by $[x] * [y] = [xy], \forall [x], [y] \in G/\rho$, and the unary operation h on G/ρ, given by $h([x]) = [f(x)], \forall\, [x] \in G/\rho$.
Proof: Firstly, we shall show that the binary operation $*$ is well-defined on G/ρ.
Let $[x], [y], [a], [b] \in G/\rho$ be arbitrary such that $[x] = [a], [y] = [b]$. Then $(x, a), (y, b) \in \rho$.
This implies that $(xy, ab) \in \rho$ (since ρ is a congruence) so that $[xy] = [ab]$,
$i.e.$, $[x] * [y] = [a] * [b]$. Therefore $*$ is well-defined on G/ρ.
Obviously, $h^2 = h$ (since $f^2 = f$).
Again $(G/\rho, *)$ is a semigroup (since (G, o) is a semigroup).
Let e be I-identity in G. Then $[e] \in G/\rho$. Now for all $[x] \in G/\rho$, we have
$[e] * [x] = [ex] = [f(x)] = h([x])$ and $[x] * [e] = [xe] = [f(x)] = h([x])$.

Let $[x] \in G/\rho$ be arbitrary. Then $x \in G$, and hence there exists $x' \in G$ such that

$$xx' = x'x = f(e)$$

Now $[x'] \in G/\rho$ and $[x] * [x'] = [xx'] = [f(e)] = h([e])$ and $[x'][x] = [x'x] = [f(e)] = h([e])$.
Therefore $(G/\rho, *, h)$ is an I-group.

Theorem 2.4 In an I-group (G, o, f), if a' be an I-inverse of $a \in G$, then $f(ab)' = f((ab)') = f(b'a'), \forall\, a, b \in G$.
Proof: Trivial.

Definition 2.3 An I-group (G, o, f) is said to be
(i) I-commutative if $f(ab) = f(ba), \forall a, b \in G$
(ii) commutative if $ab = ba, \forall a, b \in G$.
(iii) strongly I-commutative if $ab = f(ba), \forall\, a, b \in G$.

Remarks 2.2 From Definition 2.3, it is obvious that $(iii) \implies (ii) \implies (i)$.

Example 2.6 In addition, if the group (G, o) be commutative in Example 2.5, then the I-group $(G, *, f)$ is also commutative.

Now we have an open problem for I-groups, stated as "*Let* (G, o) *be a semigroup and* $f: G \to G$ *be an idempotent mapping. Let there exists* $e \in G$ *such that* $ea = f(a), \forall a \in G$, *and for every* $a \in G$, *there exists* $a' \in G$ *such that* $a'a = f(e)$. *Then* (G, o, f) *is an I-group*."

Now we have some more results given below.

Theorem 2.5 Let (G, o) be a semigroup and $f: G \to G$ be such that $f^2 = f$. Let there exists $e \in G$ such that $ea = ae = f(a), \forall a \in G$, and for every $a \in G$, there exists $a' \in G$ such that $a'a = f(e)$. Then (G, o, f) is an I-group.

Proof: Let $a \in G$ be arbitrary. Then $a'a = f(e)$ for some $a' \in G$ (1)

Now $f(a')a = (ea')a = e(a'a) = ef(e)$ (by (1)) $= f^2(e) = f(e)$ (2)

Again $f(a)f(e) = f(a)(ee) = (f(a)e)e = f^2(a)e = f(a)e = f^2(a) = f(a)$ (3)

And $f(e)f(a) = (ee)f(a) = e\big(ef(a)\big) = ef^2(a) = ef(a) = f^2(a) = f(a)$ (4)

Now $af(a') = af^2(a') = a\big(ef(a')\big) = (ae)f(a') = f(a)f(a')$ (5)

$= (ea)f(a') = e\big(af(a')\big) = f(af(a'))$ (6)

Therefore $af(a') = f(a)f(a') = f(af(a'))$ (by (5), (6)) (7)

Now, by the given condition, there exists $a'' \in G$ such that $a''a' = f(e)$ (8)

Therefore, $f(a'')f(a') = (ea'')(a'e) = e(a''a')e = (ef(e))e$ (by (8))

$= f^2(e)e = f(e)e = f^2(e) = f(e)$ (9)

Therefore $af(a') = f(a)f(a')$ (by (5)) $= \big(f(e)f(a)\big)f(a')$ (by (4))

$= f(e)(f(a)f(a') = (f(a'')f(a'))(f(a)f(a'))$ (by (9))

$= f(a'')\big(f(a')f(a)\big)f(a')$ (10)

Now $f(a')f(a) = (ea')(ae) = e(a'a)e = \big(ef(e)\big)e = f^2(e)e = f(e)e = f^2(e) = f(e)$ (11)

Using (11) in (10) we get

$af(a') = f(a)f(a') = f(a'')\big(f(e)f(a')\big) = f(a'')\Big(e\big(ef(a')\big)\Big) = f(a'')(ef^2(a'))$

$= f(a'')\big(ef(a')\big) = f(a'')f^2(a') = f(a'')f(a') = f(e)$, by (9). (12)

Therefore $f(a')a = af(a') = f(e)$, by (2) and (12)

Therefore (G, o, f) is an I-group.

Definition 2.4 Let (G, o, f) be an I-group and *H* be a nonempty subset of *G*. If *H* forms an I-group under the induced operations 'o' and *f*, we say that (H, o, f) is an I-subgroup of *G*.

Example 2.7 Consider the I-group (G, o, f) of Example 2.1 and consider the set *H* of all nonzero rational numbers. Then (H, o, f) is an I-subgroup of *G*.

Theorem 2.6 Let (G, o, f) be an I-group and *H* be a nonempty subset of *G*. Then (H, o, f) is an I-subgroup of *G* if and only if

(i) $f(H) \subseteq H$,

(ii) $ab \in H, \forall a, b \in H$,

(iii) for every $a \in H$, there exists $a' \in H$ such that $aa' = a'a = f(e)$, e being I-identity in *G*.

Proof: Let (H, o, f) be an I-subgroup of *G*. Then clearly (i), (ii) and (iii) hold.

Conversely let *H* be a nonempty subset of *G* such that (i), (ii) and (iii) hold.

Now (i) implies that the restriction of f from G to H is an idempotent mapping from H to H.
Again (ii) implies that (H, o) is a subsemigroup of the semigroup (G, o).
From (ii) and (iii) we get $f(e) \in H$.
Now for all $a \in H$ we have $af(e) = a(ee) = (ae)e = f(a)e = f^2(a) = f(a)$,
and $f(e)a = (ee)a = e(ea) = ef(a) = f^2(a) = f(a)$.
Therefore $f(e)$ is I-identity in H. Therefore (H, o, f) is itself an I-group so that it is an I-subgroup of G.

Definition 2.5 An I-subgroup H of an I-group (G, o, f) is said to be an I-normal I-subgroup of G if $f(xH) = f(Hx), \forall x \in G$, where $xH = \{xh : h \in H\}$ is a left coset of H in G and $Hx = \{hx : h \in H\}$ is a right coset of H in G.

Theorem 2.7 Let (G, o, f) be an I-group. Then the centre $Z(G) = \{x \in G : xg = gx, \forall g \in G\}$ is an I-normal I-subgroup of G.

Proof: Trivial.

Theorem 2.8 Let H be an I-subgroup of an I-group (G, o, f). Then
(i) $f(e) \in H, e$ being I-identity in G,
(ii) $f(aH) \subseteq aH, \forall a \in G$,
(iii) $aH = bH \Longrightarrow f(b'a) \in H \Longrightarrow f(aH) = f(bH)$, where b' is an I-inverse of b.

Proof: (i) Let $x \in H$. Then there exists an I-inverse x' of x in H so that $f(e) = xx' \in H$.

(ii) Firstly, we shall show that $f(H) \subseteq H$. Let $f(x) \in f(H)$ be arbitrary, where $x \in H$.
Now $f(e) \in H$. Therefore $f(x) = f^2(x) = f(x)e = (xe)e = x(ee) = xf(e) \in H$.
Therefore $f(H) \subseteq H$ (13)

Let$f(ah) \in f(aH)$ be arbitrary. Then $h \in H$.
Now $f(ah) = (ah)e = a(he) = af(h) \in af(H) \subseteq aH$, by (13),
Therefore $f(aH) \subseteq aH, \forall a \in G$.

(iii) Let $aH = bH$.
Now $f(a) = f^2(a) = f(a)e = (ae)e = a(ee) = af(e) \in aH\ (= bH)$.
Therefore $f(a) = bh$ for some $h \in H$.
$\Longrightarrow b'f(a) = (b'b)h \Longrightarrow f(b'a) = f(e)h = f(h) \in f(H)$.

Now let $f(b'a) \in f(H)$. Then $f(b'a) = f(h)$ for some $h \in H$ (14)
Let $x = f(ah_1) \in f(aH)$ be arbitrary. Then $h_1 \in H$.
From (14) we get $b'a \in f^{-1}(f(h))$ so that $b'a = p$ for some $p \in G$ such that $f(p) = f(h)$.
This implies that $bb'a = bp$.
$\Longrightarrow f(e)a = f(a) = bp \Longrightarrow f(a)h_1 = bph_1 \Longrightarrow f(ah_1) = bph_1 \Longrightarrow f(ah_1) = f(bph_1)$
$\Longrightarrow f(ah_1) = f(b)f(p)f(h_1) = f(b)f(h)f(h_1) = f\big(b(hh_1)\big) \in f(bH)$.
Therefore $f(aH) \subseteq f(bH)$.
Similarly, we can prove that $f(bH) \subseteq f(aH)$.
Therefore $f(aH) = f(bH)$.

Definition 2.6 Let (G, o, f) be an I-group. An element $a \in G$ is said be of finite order if there exists $k \in \mathbb{N}$ such that $f\big(a^k\big) = f(e), e$ being I-identity in G. In this case, the smallest positive integer n for which $f(a^n) = f(e)$, is called the I-order of a. If for all $n \in \mathbb{N}, f(a^n) \neq f(e)$, then a is said to be of infinite I-order. I-order of a is denoted by $o(a)$.

Example 2.8 In Example 2.2 $o(n) = 1, o(n-1) = n$.

Definition 2.7 Let (G_1, o, f) and $(G_2, *, h)$ be two I-groups. A mapping $\phi: G_1 \to G_2$ is said to be
(i) I-injective if for all $x, y \in G_1, (h\phi)(x) = (h\phi)(y) \Longrightarrow f(x) = f(y)$.
(ii) a homomorphism if $\phi(aob) = \phi(a) * \phi(b), \forall a, b \in G_1$.
(iii) an I-monomorphism if ϕ is a homomorphism and I-injective.
(iv) an epimorphism if ϕ is a homomorphism and surjective.
(v) an I-isomorphism if if ϕ is a homomorphism, I-injective and surjective.

Theorem 2.9 Let (G_1, o, f) and $(G_2, *, h)$ be two I-groups and $\phi: G_1 \to G_2$ is a homomorphism. Then
(i) $(h\phi f)(e_1) = h(e_2)$, where e_1 and e_2 are I-identities in G_1 and G_2 respectively.
(ii) $(h\phi)(a')$ is an I-inverse of $(h\phi)(a)$, where a' is an I-inverse of a, for all $a \in G_1$.
(iii) for all $a \in G_1, \phi(a^n) = \left(\phi(a)\right)^n, \forall\, n \in \mathbb{N}$.
(iv) for $a \in G_1$, if $o(a)$ is finite, then$o((h\phi f)(a))$ divides $o(a)$.

Proof: (i) $f(e_1)of(e_1) = f(e_1) \Longrightarrow (\phi f)(e_1) * (\phi f)(e_1) = (\phi f)(e_1)$
$\Longrightarrow (h\phi f)(e_1) * (h\phi f)(e_1) = (h\phi f)(e_1) = (h\phi f)(e_1) * e_2 \Longrightarrow (h\phi f)(e_1) = h(e_2)$.

(ii) $aa' = f(e_1) \Longrightarrow (\phi f)(e_1) = \phi(a)\phi(a') \Longrightarrow (h\phi f)(e_1) = (h\phi)(a)(h\phi)(a') = h(e_2)$, by (i).
Again $a'a = f(e_1) \Longrightarrow (\phi f)(e_1) = \phi(a')\phi(a) \Longrightarrow (h\phi f)(e_1) = (h\phi)(a')(h\phi)(a) = h(e_2)$, by (i).
$\Longrightarrow (h\phi)(a')$ is an I-inverse of $(h\phi)(a)$ in G_2.

(iii) Trivial.

(iv) Let $o(a) = n$ be finite so that $f(a^n) = f(e_1)$.
Now $((h\phi f)(a))^n = (h\phi f)(a^n) = (h\phi f)(e_1) = h(e_2)$ which is an I-identity in G_2.
This implies that $h(((h\phi f)(a))^n) = h(e_2)$.
Therefore $o((h\phi f)(a))$ is finite and $o\left((h\phi f)(a)\right) \leq n$. Let $o\left((h\phi f)(a)\right) = k$.
Then $h\left(((h\phi f)(a))^k\right) = ((h\phi f)(a))^k = h(e_2)$.
Let $n = pk + r$, where p is a nonnegative integer and $0 \leq r < k$.
Now $h(((h\phi f)(a))^r) = ((h\phi f)(a))^r = ((h\phi f)(a))^n * \left(\left((h\phi f)(a)\right)^k\right)^{-p} = h(e_2) * \left(h(e_2)\right)^{-p}$
$= h(e_2)$.
This implies that $r = 0$ so that k divides n.

Theorem 2.10 Let (G, o, f) be an I-group and H be an I-normal I-subgroup of G. Then $f(H), f^{-1}(H)$ are I-normal I-subgroups of G.

Proof: Here $f(xH) = f(Hx), \forall x \in G$. Clearly $f(H)$ is a nonempty subset of G and
$f\left(f(H)\right) = f(H) \subseteq f(H)$. Let $x = f(a), y = f(b) \in f(H)$ be arbitrary, where $a, b \in H$.
Then $xy = f(a)f(b) = f(ab) \in f(H)$, since $a, b \in H \Longrightarrow ab \in H$.
Now there exists $a' \in H$ such that $aa' = a'a = f(e), e$ being I-identity in G so that $f(e)$ is I-identity in H.
This implies that $xa' = f(a)a' = f(aa') = f(e) = f(a'a) = a'f(a) = a'x$.
Therefore $f(H)$ is an I-subgroup of G.
Now it can be easily proved that $x\left(f(H)\right) = f(xH)$ and $\left(f(H)\right)x = f(Hx), \forall\, x \in G$.
Therefore $f(xH) = f(Hx), \forall\, x \in G \Longrightarrow x\left(f(H)\right) = \left(f(H)\right)x, \forall\, x \in G$.
Therefore $f(H)$ is an I-normal I-subgroup of G.

Since $f(e) \in H$, hence $e \in f^{-1}(H)$ so that $f^{-1}(H)$ is a nonempty subset of G.
Since $f(H) \subseteq H$, hence $H \subseteq f^{-1}(H)$. Therefore $f\left(f^{-1}(H)\right) \subseteq H \subseteq f^{-1}(H)$.

Let $x, y \in f^{-1}(H)$ be arbitrary. Then $f(x), f(y) \in H$. This implies that $f(xy) = f(x)f(y) \in H$, so that $xy \in f^{-1}(H)$.

Now $f(x) \in H \Rightarrow f(x) = f^2(x) \in f(H)$. Also $f(H)$ is an I-subgroup of G. Therefore, there exists $x' = f(x'') \in f(H)$, where $x'' \in H$, such that $f(x)x' = x'f(x) = f(e)$ (15)

Now $xx' = xf(x'') = f(x)f(x'') = f(x)x' = f(e)$ (by (15)),

and $x'x = f(x'')x = f(x'')f(x) = x'f(x) = f(e)$ (by (15)).

Therefore $xx' = x'x = f(e)$. Now $x' \in f(H) \subseteq H \subseteq f^{-1}(H)$.

Therefore $f^{-1}(H)$ is an I-subgroup of G.

Now we shall show that $f(xf^{-1}(H)) = f(f^{-1}(H)x), \forall x \in G$.

Let $p = f(xa) \in f(xf^{-1}(H))$ be arbitrary, where $a \in f^{-1}(H)$ so that $f(a) \in H$.

Now $p = f(xa)f(x)f(a) \in f(x)H = f(xH) = f(Hx)$.

Therefore $p = f(hx)$ for some $h \in H$.

Therefore $p = f(hx) \in f(f^{-1}(H)x)$, since $h \in H \subseteq f^{-1}(H)$.

Therefore $f(xf^{-1}(H)) \subseteq f(f^{-1}(H)x)$.

Similarly, we can prove that $f(f^{-1}(H)x) \subseteq f(xf^{-1}(H))$.

Therefore $f(xf^{-1}(H)) = f(f^{-1}(H)x), \forall x \in G$.

Therefore $f^{-1}(H)$ is an I-normal I-subgroup of G.

Theorem 2.11 Let (G_1, o, f) and $(G_2, *, h)$ be two I-groups and $\phi: G_1 \to G_2$ is a homomorphism.

(i) If H be an I-subgroup of G_1then $\phi(H)$ is an I-subgroup of G_2, provided $(h\phi)(H) \subseteq \phi(H)$.
In addition, if H is I-normal in G_1 and ϕ is surjective, then $\phi(H)$ is normal in G_2.

(ii) If H be an I-subgroup of G_1then $(h\phi)(H)$ is an I-subgroup of G_2.
In addition, if H is I-normal in G_1 and ϕ is surjective, then $(h\phi)(H)$ is normal in G_2.

(iii) If K be an I-normal I-subgroup of G_2then $(\phi^{-1}h^{-1})(K)$ is an I-normal I-subgroup of G_1.

Proof: Let e_1 and e_2 are I-identities in G_1 and G_2 respectively.

(i) Let$h\big(\phi(H)\big) \subseteq \phi(H)$ (16)

Clearly, $\phi(H)$ is a nonempty subset of G_2. Let $x = \phi(a), y = \phi(b) \in \phi(H)$ be arbitrary, where $a, b \in H$. Then $ab \in H$ so that $xy = \phi(a)\phi(b) = \phi(ab) \in \phi(H)$.

Now $a \in H \Rightarrow$ there exists $a' \in H$ such that $aa' = a'a = f(e_1)$ (17)

Now $x * (h\phi)(a') = \phi(a) * (h\phi)(a') = h\big(\phi(a) * \phi(a')\big) = (h\phi)(aa') = (h\phi f)(e_1)$ (by (17)).

Therefore $x * (h\phi)(a') = h(e_2)$.

Again $(h\phi)(a') * x = (h\phi)(a') * \phi(a) = h(\phi(a') * \phi(a)) = (h\phi)(a'a) = (h\phi f)(e_1)$ (by (17)).

Therefore $(h\phi)(a') * x = h(e_2)$.

Therefore $x * (h\phi)(a') = (h\phi)(a') * x = h(e_2)$, and $(h\phi)(a') \in (h\phi)(H) \subseteq \phi(H)$.

Therefore $\phi(H)$ is an I-subgroup of G_2.

Now let H be I-normal in G_1 and ϕ is surjective. Then $f(xH) = f(Hx), \forall\, x \in G_1$ (18)

We shall show that $h\big(y\phi(H)\big) = h(\phi(H)y), \forall\, y \in G_2$. Let $y \in G_2$ be arbitrary. Since ϕ is surjective, there exists $x \in G_1$ such that $y = \phi(x)$.

Let $p = h\big(y\phi(a)\big) \in h(y\phi(H))$ be arbitrary. Then $a \in H$.

Now $p = h\big(\phi(x)\phi(a)\big) = h\big(\phi(xa)\big) = \phi(xa)h(e_2) = \phi(xa)(h\phi f)(e_1) = h(\phi(xa)(\phi f)(e_1))$
$= (h\phi)\big(xaf(e_1)\big) = (h\phi)\big(f(xa)\big) = (h\phi)(f(bx))$ (by (18) $f(xa) = f(bx)$ for some $b \in H$)
$= (h\phi)\big(bxf(e_1)\big) = h\big(\phi(bx)(\phi f)(e_1)\big) = \phi(bx)(h\phi f)(e_1) = \phi(bx)h(e_2) = (h\phi)(bx)$
$= (h\phi)(b)(h\phi)(x) = (h\phi)(b)h(y) = h(\phi(b)y) \in h(\phi(H)y)$.

Therefore $h\big(y\phi(H)\big) \subseteq h(\phi(H)y)$.

Similarly, we shall get $h(\phi(H)y) \subseteq h(y\phi(H))$.

Therefore $h(\phi(H)y) = h(y\phi(H))$, so that $\phi(H)$ is I-normal in G_2.

(ii) Here $h\phi$ is a homomorphism from G_1 to G_2 and $h\big((h\phi)(H)\big) = (h\phi)(H)$. Therefore, by the first part of (i) of this theorem, we can say that $(h\phi)(H)$ is an I-subgroup of G_2.

Second part is similar to that of (i) of this theorem.

(iii) Let K be an I-subgroup of G_2. Since $h(e_2) \in K$ and $(h\phi f)(e_1) = h(e_2)$, hence $(h\phi f)(e_1) = h(e_2) = h^2(e_2) \in h(K) \subseteq K$. This implies that $f(e_1) \in (\phi^{-1}h^{-1})(K)$ so that $(\phi^{-1}h^{-1})(K) \neq \Phi$.

Now we shall show that $f(\phi^{-1}h^{-1}(K)) \subseteq \phi^{-1}h^{-1}(K)$.

Let $p = f(x) \in f(\phi^{-1}h^{-1}(K))$ be arbitrary, where $x \in \phi^{-1}h^{-1}(K)$. Then $(h\phi)(x) \in K$. (19)

Now $(h\phi)(x) = \phi(x)h(e_2) = \phi(x)(h\phi f)(e_1) = h\big(\phi(x)(\phi f)(e_1)\big) = (h\phi)\big(xf(e_1)\big) = (h\phi f)(x)$

Therefore $(h\phi f)(x) = (h\phi)(x) \in K$, by (19). This implies that $p = f(x) \in \phi^{-1}h^{-1}(K)$.

Therefore $f(\phi^{-1}h^{-1}(K)) \subseteq \phi^{-1}h^{-1}(K)$.

Let $x, y \in \phi^{-1}h^{-1}(K)$ be arbitrary. Then $\phi(x), \phi(y) \in h^{-1}(K)$ so that $\phi(xy) = \phi(x)\phi(y) \in h^{-1}(K)$ (by Theorem 2.10 $h^{-1}(K)$ is an I-subgroup of G_2).

Therefore $xy \in \phi^{-1}h^{-1}(K)$.

Let $x \in \phi^{-1}h^{-1}(K)$ be arbitrary. Then $\phi(x) \in h^{-1}(K)$. Since $h^{-1}(K)$ is an I-subgroup of G_2hence an I-inverse zof $\phi(x)$ exists in $h^{-1}(K)$.

Now $\phi(x)\phi(x') = \phi(xx') = (\phi f)(e_1)$, $\phi(x')\phi(x) = \phi(x'x) = (\phi f)(e_1)$, and $(h\phi f)(e_1) = h(e_2)$ imply that $(\phi f)(e_1)$ is an I-identity in G_2.

Again, $(h\phi f)(e_1) = h(e_2) \in K$ (Since K is an I-subgroup of G_2) implies that $(\phi f)(e_1) \in h^{-1}(K)$.

Therefore $h(z) = (h\phi)(x')$ (20)

Now $(h\phi)(x) \in K$ and K is an I-subgroup of G_2 implies that $(h\phi)(x') \in K$, by (20).

Therefore $x' \in (\phi^{-1}h^{-1})(K)$. Therefore, $(\phi^{-1}h^{-1})(K)$ is an I-subgroup of G_1.

Now let K be I-normal in G_2 so that $h(aK) = h(Ka), \forall\, a \in G_2$ (21)

We shall show that $f\big(x((\phi^{-1}h^{-1})(K))\big) = f\big(((\phi^{-1}h^{-1})(K))x\big), \forall x \in G_1$.

Let $p = f(xy) \in f\big(x((\phi^{-1}h^{-1})(K))\big)$ be arbitrary, where $y \in (\phi^{-1}h^{-1})(K)$ so that $(h\phi)(y) \in K$ (22)

Now $f(xy) = xyf(e_1)$.

Therefore $(h\phi f)(xy) = (h\phi)(xy)(h\phi f)(e_1) = (h\phi)(xy)h(e_2) = (h\phi)(xy) = h(\phi(x)\phi(y))$

$= (h\phi)(x)(h\phi)(y) = (h\phi)(x)k_1$, where $(h\phi)(y) = k_1 \in K$, by (22)

$= h(\phi(x)k_1) == h(k_2\phi(x))$, taking $a = \phi(x)$ in (21).

$= k_2(h\phi)(x) \in K(h\phi)(x)$

$\Rightarrow f(xy) \in (\phi^{-1}h^{-1})(K)x \;\Rightarrow f(xy) = f^2(xy) \in f\big(((\phi^{-1}h^{-1})(K))x\big)$.

Therefore $f\big(x((\phi^{-1}h^{-1})(K))\big) \subseteq f\big(((\phi^{-1}h^{-1})(K))x\big)$.

Similarly, $f\big(((\phi^{-1}h^{-1})(K))x\big) \subseteq f\big(x((\phi^{-1}h^{-1})(K))\big)$.

Therefore $f\big(x((\phi^{-1}h^{-1})(K))\big) = f\big(((\phi^{-1}h^{-1})(K))x\big), \forall x \in G_1$.

This completes the proof.

Definition 2.8 Let $\phi: G_1 \to G_2$ be a homomorphism, where(G_1, o, f) and $(G_2, *, h)$ are two I-groups. The Kernel of ϕ is defined by $Ker\phi = \{a \in G_1 : (h\phi)(a) = h(e_2)\}$, where e_2 is I-identity in G_2.

Theorem 2.12 Let $\phi: G_1 \to G_2$ be a homomorphism, where(G_1, o, f) and $(G_2, *, h)$ are two I-groups. Then $Ker\phi$ is an I-normal I-subgroup of G_1.

Proof: Let e_1 and e_2 be I-identities in G_1 and G_2 respectively. Since $f(e_1) \in G_1$ and $(h\phi f)(e_1) = h(e_2)$, hence $f(e_1) \in Ker\phi$ so that $Ker\phi$ is a nonempty subset of G_1.

Now we shall show that $f(Ker\phi) \subseteq Ker\phi$. Let $y = f(x) \in f(Ker\phi)$ be arbitrary, where $x \in Ker\phi$.

Clearly $y = f(x) \in G_1$ and $(h\phi)(x) = h(e_2)$.

Now $(h\phi)(y) = (h\phi)\big(f(x)\big) = (h\phi)\big(xf(e_1)\big) = (h\phi)(x)(h\phi f)(e_1) = h(e_2)h(e_2) = h(e_2)$ so that $y \in Ker\phi$. Therefore $f(Ker\phi) \subseteq Ker\phi$.

Let $x, y \in Ker\phi$ be arbitrary. Then $(h\phi)(x) = h(e_2), (h\phi)(y) = h(e_2)$ (23)

Clearly $xy \in G_1$. Now $(h\phi)(xy) = (h\phi)(x)(h\phi)(y) = h(e_2)h(e_2) = h(e_2)$. (by (23))

Therefore $xy \in Ker\phi$.

Let $x \in Ker\phi$ be arbitrary. Then $(h\phi)(x) = h(e_2)$ (24)

Now $x^{'} \in G_1$ such that $xx^{'} = f(e_1)$. This implies that $(h\phi)(xx^{'}) = (h\phi f)(e_1)$.

Therefore $(h\phi)(x)(h\phi)(x^{'}) = h(e_2)$.

$\Longrightarrow h(e_2)(h\phi)(x^{'}) = h(e_2)$, by (24) $\Longrightarrow (h\phi)(x^{'}) = h(e_2) \Longrightarrow x^{'} \in Ker\phi.$

Therefore $Ker\phi$ is an I-subgroup of G_1.

Let $H = Ker\phi$. Now we shall show that $f(xH) = f(Hx), \forall x \in G_1$.

Let $p = f(xy) \in f(xH)$ be arbitrary, where $y \in H$ so that $(h\phi)(y) = h(e_2)$ (25)

Now $p = f(xy) = xyf(e_1) = xyf^2(e_1) = xyf(x^{'}x) = f\big((xyx^{'})x\big) \in f(Hx)$, since

$(h\phi)(xyx^{'}) = (h\phi)(x)(h\phi)(y)(h\phi)(x^{'}) = (h\phi)(x)h(e_2)(h\phi)(x^{'})$, by (25)

$= (h\phi)(x)(h\phi)(x^{'}) = (h\phi)(xx^{'}) = (h\phi f)(e_1) = h(e_2) \Longrightarrow xyx^{'} \in H.$

Therefore $f(xH) \subseteq f(Hx)$.

Similarly, $f(Hx) = f(xH)$.

Therefore $f(xH) = f(Hx), \forall x \in G_1$.

Hence the result.

Theorem 2.13 Let $\phi: G_1 \to G_2$ be a homomorphism, where(G_1, o, f) and $(G_2, *, h)$ are two I-groups. ϕ is I-injective iff $Ker\phi \subseteq Kerf = \{x \in G_1 : f(x) = f(e_1)\}$, where e_1 is I-identity in G_1.

Proof: Let e_1 and e_2 be I-identities in G_1 and G_2 respectively. Let ϕ be I-injective. Let $x \in Ker\phi$. Then $(h\phi)(x) = h(e_2) = (h\phi)(f(e_1))$. This implies that $f(x) = f(e_1)$ (Since ϕ is I-injective).

Therefore $x \in Kerf$, so that $Ker\phi \subseteq Kerf$.

Conversely, let $Ker\phi \subseteq Kerf$. Let $x, y \in G_1$ be arbitrary such that $(h\phi)(x) = (h\phi)(y)$ (26)

Now, $(h\phi)(xy^{'}) = (h\phi)(x)(h\phi)(y^{'}) = (h\phi)(y)(h\phi)(y^{'})$, by (26)

$= (h\phi)(yy^{'}) = (h\phi)\big(f(e_1)\big) = h(e_2)$.

Therefore $xy^{'} \in Ker\phi \subseteq Kerf$.

This implies that $f(xy^{'}) = f(e_1)$.

$\Longrightarrow f(xy^{'})f(y) = f(e_1)f(y) \Longrightarrow f(xy^{'}y) = f(y) \Longrightarrow f\big(xf(e_1)\big) = f\big(f(x)\big) = f(x) = f(y).$

Therefore ϕ is I-injective.

Theorem 2.14 Let $\phi: G_1 \to G_2$ be an I-isomorphism, where(G_1, o, f) and $(G_2, *, h)$ are two I-groups.

(i) $o(a) = o((\phi f)(a))$, for every $a \in G_1$.

(ii) If G_2 is strongly I-commutative then G_1 is I-commutative.

(iii) If G_2 is commutative then G_1 is I-commutative.

(iv) If G_2 is I-commutative then G_1 is I-commutative.

Proof: (i) Let $o(a) = n$ be finite. Then $f(a^n) = f(e_1)$.
Now $h(((\phi f)(a))^n) = (h\phi)(f(a^n)) = (h\phi f)(e_1) = h(e_2)$.
This implies that $o((\phi f)(a))$ is finite. Let $o\big((\phi f)(a)\big) = m$. Then $m \leq n$.
Now $h(((\phi f)(a))^m) = h(e_2) \Rightarrow (h\phi)\big(f(a^m)\big) = (h\phi)\big(f(e_1)\big) \Rightarrow f(a^m) = f(e_1)$, since ϕ is I-injective.
Therefore $n \leq m$. Hence $m = n$.

Let $o(a)$ be infinite. If possible, let $o((\phi f)(a))$ be finite. Then by the second part of the proof of this theorem we shall get $o(a)$ is finite which is a contradiction.
Therefore $o((\phi f)(a))$ is infinite.

(ii) Let G_2 be strongly I-commutative.
Now, $a, b \in G_1 \Rightarrow \phi(a), \phi(b) \in G_2 \Rightarrow \phi(a)\phi(b) = h\big(\phi(b)\phi(a)\big) \Rightarrow \phi(ab) = (h\phi)(ba)$
$\Rightarrow (h\phi)(ab) = (h\phi)(ba) \Rightarrow f(ab) = f(ba)$, since ϕ is I-injective.
Therefore G_1 is I-commutative.

(iii) and (iv) are similar to (ii).

Theorem 2.15 Let H be an I-subgroup of an I-group (G, o, f). Then for any two left cosets xH and yH of H in G
(i) there is an I-bijection, *i.e.*, an I-injection and surjection between xH and yH.
(ii) either $f(xH) = f(yH)$ or $xH \cap yH = \Phi$.

Proof: (i) Define $g: xH \to yH$ by $g(xh) = yh, \forall h \in H$. Let $xa, xb \in xH$ be arbitrary such that $(fg)(xa) = (fg)(xb)$.
This implies that $f(ya) = f(yb)$.
$\Rightarrow yf(a) = yf(b) \Rightarrow f(a) = f(b) \Rightarrow xf(a) = xf(b) \Rightarrow f(xa) = f(xb)$.
Therefore g is I-injective. Clearly, g is surjective. Therefore g is an I-bijection.

(ii) We can say that either $xH \cap yH = \Phi$ or $xH \cap yH \neq \Phi$. Let $xH \cap yH \neq \Phi$. Let $z \in xH \cap yH$. Then $z = xa = yb$ for some $a, b \in H$.
This implies that $y'xaa' = y'yba'$
$\Rightarrow y'xf(e) = f(e)ba'$, e being I-identity in G.
$\Rightarrow f(y'x) = f(ba') \in f(H) \Rightarrow f(xH) = f(yH)$.

Note 2.1 Let H be an I-normal I-subgroup of an I-group (G, o, f). Define a binary relation ρ on G as:
$\rho = \{(a, b) \in G \times G : f(aH) = f(bH)\}$. Then ρ is an equivalence relation on G, so that we have the quotient set G/ρ. Now for any $\bar{x} \in G/\rho$, it can be shown that $\bar{x} = f(\bar{x}) = \overline{f(x)} = f(xH) \subseteq xH$.
Define $\bar{x} * \bar{y} = \overline{xoy}, \forall \bar{x}, \bar{y} \in G/\rho$. Let $\bar{x}, \bar{y}, \bar{a}, \bar{b} \in G/\rho$ such that $\bar{x} = \bar{a},\ \bar{y} = \bar{b}$.
Then $f(xH) = f(aH), f(yH) = f(bH)$. This implies that $f(x) \in aH, f(y) \in bH$, so that $f(x) = ah_1, f(y) = bh_2$ for some $h_1, h_2 \in H$.
We claim that $\bar{x} * \bar{y} = \bar{a} * \bar{b}, i.e., \overline{xoy} = \overline{aob}, i.e., f\big((xy)H\big) = f((ab)H)$.
For any $xyh \in (xy)H$, we have
$f(xyh) = f(ab)f((ab)')xyh = f(a)f(b)f(b')f(a')f(x)f(y)h$
$= f(a)f(b)f(b')f(a')ah_1f(y)h = f(a)f(b)f(b')h_1f(y)h = f(a)f(b)f(b')f(y)h_3h$, since H is I-normal, $h_1f(y) = f(h_1y) \in f(Hy) = f(yH)$ so that $h_1f(y) = f(yh_3) = f(y)h_3$ for some $h_3 \in H$.
Therefore $f(xyh) = f(a)f(b)f(b')bh_2h_3h = f(ab)h_2h_3h = f(abh_2h_3h) \in f((ab)H)$.
Therefore $f\big((xy)H\big) \subseteq f((ab)H)$.
Similarly, $f\big((ab)H\big) \subseteq f\big((xy)H\big)$. Therefore $f\big((xy)H\big) = f((ab)H)$, so that $*$ is well-defined.
Clearly, G/ρ is closed with respect to $*$.

Now for all $\bar{x}, \bar{y}, \bar{z} \in G/\rho$, we have

$(\bar{x} * \bar{y}) * \bar{z} = \overline{xy} * \bar{z} = \overline{(xy)z} = \overline{x(yz)} = \bar{x} * \overline{yz} = \bar{x} * (\bar{y} * \bar{z})$ so that $*$ is associative.

We see that I-identity $e \in G$ so that $\bar{e} \in G/\rho$. Now for all $\bar{x} \in G/\rho$, we have

$\bar{x} * \bar{e} = \overline{xe} = \overline{f(x)} = f(\bar{x}) = \bar{x}$ and $\bar{e} * \bar{x} = \overline{ex} = \overline{f(x)} = f(\bar{x}) = \bar{x}$.

Therefore $\bar{e}$ is identity in G/ρ.

Let $\bar{x} \in G/\rho$ be arbitrary. Then $\overline{x'} \in G/\rho$ and $xx' = x'x = f(e)$.

Now $\bar{x} * \overline{x'} = \overline{xx'} = \overline{f(e)} = f(\bar{e}) = \bar{e}$ and $\overline{x'} * \bar{x} = \overline{x'x} = \overline{f(e)} = f(\bar{e}) = \bar{e}$.

Therefore $(G/\rho, *)$ is a group so that $(G/\rho, *, I)$ is an I-group, where I is the identity mapping on G/ρ. G/ρ is denoted by G/H.

Theorem 2.16 Let H be an I-normal I-subgroup of an I-group (G, o, f). Define $\theta: (G, o, f) \to (G/\rho, *, I)$by $\theta(x) = \bar{x}, \forall x \in G$. Then θ is an epimorphism.

Proof: Trivial.

Theorem 2.17 Let $\phi: G_1 \to G_2$ be an epimorphism, where (G_1, o, f) and $(G_2, *, h)$ are two I-groups. Let $H = Ker\phi$. Consider the epimorphism $\theta: (G_1, o, f) \to (G_1/H, *, I)$, given by $\theta(x) = \bar{x}, \forall x \in G_1$. Then there exists an I-isomorphism$\psi: G_1/H \to G_2$ such that $\phi = \psi\theta$.

Proof: Let e_1 and e_2 be I-identities in G_1 and G_2 respectively.

For any $a \in G_1$, $\theta(a) = \bar{a} = \{b \in G_1 : f(aH) = f(bH)\}$.

Define $\psi: G_1/H \to G_2$ by $\psi(\bar{a}) = (h\phi)(a), \forall \bar{a} \in G_1/H$.

Let $a, b \in G_1$ be arbitrary such that $\bar{a} = \bar{b}$. Then $f(aH) = f(bH)$.

This implies that $f(a)H = f(b)H$, since $f(aH) = f(a)H$.

$\Longrightarrow f(b'a) \in f(H) \subseteq H = Ker\phi$, where b' is an I-inverse of b.

$\Longrightarrow (h\phi)\big(f(b'a)\big) = h(e_2) = (h\phi f)(e_1) \Longrightarrow (h\phi)(b'a)(h\phi f)(e_1) = (h\phi f)(e_1)$

$\Longrightarrow (h\phi)(b'a)h(e_2) = h(e_2) \Longrightarrow (h\phi)(b'a) = h(e_2) \Longrightarrow (h\phi)(bb'a) = (h\phi)(b)$

$\Longrightarrow (h\phi)(f(e_1)a) = (h\phi)(b) \Longrightarrow (h\phi f)(e_1)(h\phi)(a) = (h\phi)(b) \Longrightarrow h(e_2)(h\phi)(a) = (h\phi)(b)$

$\Longrightarrow (h\phi)(a) = (h\phi)(b) \Longrightarrow \psi(\bar{a}) = \psi(\bar{b}) \Longrightarrow \psi$ is well-defined.

Clearly, ψ is a homomorphism.

Let $\bar{a}, \bar{b} \in G/H$ be such that $(h\psi)(\bar{a}) = (h\psi)(\bar{b})$.

This implies that $(h\phi)(a) = (h\phi)(b) \Longrightarrow (h\phi)(b'a) = (h\phi)(b'b)$

$\Longrightarrow (h\phi)(b'a)(h\phi f)(e_1) = (h\phi f)(e_1) \Longrightarrow (h\phi f)(b'a) = (h\phi f)(e_1) = h(e_2)$

$\Longrightarrow f(b'a) \in Ker\phi = H \Longrightarrow f(aH) = f(bH) \Longrightarrow \bar{a} = \bar{b}$.

Therefore ψ is injective.

Since ϕ is surjective, hence ψ is surjective.

Therefore ψ is an isomorphism.

Now for all $a \in G_1$, we have $(\psi\theta)(a) = \psi(\bar{a}) = (h\phi)(a) \Longrightarrow \psi\theta = h\phi$.

Conclusion

In this paper, we have studied some basic properties of I-groups. A lot of opportunity is there for further studies.

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