Ginzburg-Landau Equation for Thermal Convection in Porous Media with Heat Source

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Abstract

The paper presents an investigation of a weakly non-linear stability analysis of thermal convection in porous media using the Lorenz model. The Ginzburg-Landau model is then obtained from the Lorenz model using which an expression for Nusselt number is obtained in closed form. It is found that heat source enhances amount of heat transport whereas heat sink diminishes the same. The influence of the thermal and mechanical anisotropies on the Nusselt number is to oppose each other in the case of both heat source and heat sink. It is observed that the anisotropic effects are prevalent only for short time whereas heat source has a sustained influence.

Key words: porous media, thermal convection, internal heat source, Ginzburg-Landau model.

1.Introduction

Understanding the convective instability in a fluid saturated porous layer subjected to various additional effects has been given a great deal of effort due to its applications in a variety of engineering and geophysical problems, energy storage applications oil recovery process in petroleum industry and so on. One of the effective mechanisms to control convective instability is maintaining a non-uniform temperature gradient across the fluid layer. Such a temperature gradient can be generated by uniformly distributed internal heat sources, injecting the fluid at one of the boundaries and removal of the fluid at the other, called throughflow and also by uniform heating or cooling at the boundaries are a few to mention.

The study of natural convection in a porous medium has been understood and well documented in the works of Kaviany (1995), Ingham and Pop (1998), Vafai (2000), Crolet (2000) and Nield and Bejan (2006). Somerton et. al., (1984) studied the natural convection in a volumetrically heated porous layer. Rees and Pop (1995) investigated free convection induced by a vertical wavy surface with uniform heat flux in a porous medium. Free convection in porous media was analyzed by Vadasz (1998). Khalili and Shivakumara (1998) investigated the onset of convection in a horizontal, isotropic porous layer including the effects of through-flow and a uniformly distributed internal heat generation for different types of hydrodynamic boundary conditions. Khalili*et al.,* (2002) studied the convective instabilities caused by a non-uniform temperature gradient due to vertical throughflow and internal heat generation in an anisotropic porous layer. Influence of Darcy number on the onset of convection in a porous layer with uniform heat source was investigated by Nouri-Borujerdi et. al. (2008). Israel-Cookey et. al. (2010) investigated the onset of thermal instability in a low Prandtl number fluid with internal heat source in a porous medium.

The study of finite amplitude convection (Veronis, 1966), using a truncated Fourier representation, has gained momentum in recent years owing to its simplicity and nonlinear complexity of the solution. We note that the study of finite amplitude Rayleigh-Bénard convection in an anisotropic porous medium with internal heat generation by means of a minimal Fourier series representation does not seem to have been undertaken. Accordingly, we study this aspect in the paper.

2. Mathematical Formulation

Consider an infinite horizontal anisotropic porous layer of a Boussinesquian, Newtonian fluid of depth'd' that supports a temperature gradient ΔT (see figure 1). The upper and lower boundaries are maintained at constant temperatures T_0 and $T_0 + \Delta T$ ($\Delta T > 0$) respectively. For mathematical tractability we confine ourselves to two-dimensional rolls so that all physical quantities are independent of *y*, a horizontal co-ordinate. Further, the boundaries are assumed to be free and perfect conductors of heat. In this paper we assume the dynamic coefficient of viscosity μ and thermal diffusivity χ to be constants. Density and heat source are assumed to be temperature-dependent in the problem along with the Boussinesq approximation. The governing equations describing the Rayleigh-Bénard instability situation in an anisotropic porous medium with constant viscosity Newtonian fluid are

$$\nabla . \vec{q} = 0, \qquad (2.1)$$

$$\rho_R\left[\frac{1}{\Phi}\frac{\partial \vec{q}}{\partial t}+\frac{1}{\Phi^2}\left(\vec{q}\cdot\nabla\right)\vec{q}\right] = -\nabla p - \rho\vec{g} - \mu_f \mathbf{k}.\vec{q} + \mu_p \nabla^2 \vec{q}, \qquad (2.2)$$

$$\frac{\partial T}{\partial t} + \vec{q}.\nabla T = \chi \; \eta\left[\; \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2}\right] Q + T\,(\,T - \;_0), \qquad (2.3)$$

where $\bar{q}$ isthe velocity vector, Φ is the porosity of the medium, μ_f is the viscosity of the fluid, μ_p is the effective viscosity, ρ_R is the reference density, $\mathbf{k} = k_x^{-1}\hat{i}\hat{i} + k_z^{-1}\hat{k}\hat{k}$ is the permeability tensor, χ_v is the thermal diffusivity in the vertical direction, η is thermal anisotropy parameter and the last term in the last equation denotes the temperature-dependent heat source(sink).

The density equation of state is

$$\rho = \rho_0\left[1-\beta\left(T-T_0\right)\right]. \qquad (2.4)$$

Taking the velocity, temperature and density fields in the quiescent basic state to be $\vec{q}_b(z) = (0,0),\ T_b(z)$ and $\rho_b(z)$, we obtain the quiescent state solution in the form:

$$\left.\begin{aligned} &\vec{q}_b = (0,\,0), \quad T_b = T_0 + \Delta T\ f\left(\frac{z}{d}\right), \\ &\rho_b\left(\frac{z}{d}\right) = \rho_0\left[1-\beta\,\Delta T\ f\left(\frac{z}{d}\right)\right],\ p_b\left(\frac{z}{d}\right) = -\int \rho_b\left(\frac{z}{d}\right) g\ d\left(\frac{z}{d}\right) + C_1, \end{aligned}\right] \qquad (2.5)$$

where $f(\frac{z}{d}) = \frac{\sin\sqrt{R_i}\,(1-z/d)}{\sin\sqrt{R_i}}$ and C_1 is the constant of integration and $R_I = \frac{Qd^2}{\chi_v}$ (Internal Rayleigh number). On the quiescent basic we superimpose perturbations in the form:

$$\vec{q} = \vec{q_b} + \vec{q}', \; T = T_b(Z) + T', \; \rho = \rho_b(Z) + \rho', \; p = p_b(Z) + p', \tag{2.6}$$

where the prime indicates a perturbed quantity. Since we consider only two-dimensional disturbances, we introduce the stream function as follows:

$$u' = \frac{\partial \psi'}{\partial z}, w' = \frac{\partial \psi'}{\partial x}, \tag{2.7}$$

which satisfy the Eq. (2.1) in the perturbed state. Eliminating the pressure in Eq. (2.2), incorporating the quiescent state solution and non-dimensionalizing the resulting equation as well as equation (2.3) using the following definition

$$(X, Z) = \left(\frac{x}{d}, \frac{z}{d}\right), \quad \tau = \frac{\chi_v}{d^2} t, \psi = \frac{\psi'}{\chi_v}, \Theta = \frac{T'}{\Delta T}, \tag{2.8}$$

we obtain the dimensionless form of the vorticity and heat transport equations as

$$\frac{1}{Pr}\frac{\partial}{\partial \tau}\left(\nabla^2 \psi\right) = \Lambda \nabla^4 \psi - R_E \frac{\partial \Theta}{\partial X} - Da^{-1}\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{1}{\varepsilon}\frac{\partial^2 \psi}{\partial z^2}\right) + \frac{1}{Pr} J\left(\psi, \nabla^2 \psi\right), \tag{2.9}$$

$$\frac{\partial \Theta}{\partial \tau} = -\frac{\partial \Psi}{\partial X}\frac{df}{dz} + \eta \frac{\partial^2 \Theta}{\partial x^2} + \frac{\partial^2 \Theta}{\partial z^2} + R_I \Theta + J(\Psi, \Theta), \tag{2.10}$$

where J is the Jacobian.

The non-dimensional parameters appearing in Eqs. (2.9) – (2.10) are defined below:

$R_E = \frac{\beta g \Delta T d^3}{\nu \chi_v}$ (External Rayleigh number), $Da^{-1} = \frac{d^2}{k_v}$ (Inverse Darcy number),

$\varepsilon = \frac{k_h}{k_v}$ (Mechanical anisotropy parameter), $\eta = \frac{\chi_h}{\chi_v}$ (Thermal anisotropy parameter),

$\Lambda = \frac{\mu_p}{\mu_f}$ (Brinkman number), $\Pr = \frac{\nu}{\chi}$ (Prandtl number) and R_I is as defined earlier.

Eqs. (2.9) – (2.10) are solved using the boundary conditions

$$\psi = \frac{\partial^2 \psi}{\partial Z^2} = \Theta = 0 \quad \text{at } Z = 0, 1. \tag{2.11}$$

In the next section, we discuss the linear stability analysis, which is of great utility in the local nonlinear stability analysis to be discussed further on.

3.Linear Stability Theory

In order to study the linear theory, we consider the linear version of Eqs. (2.9) – (2.10) and assume the solutions to be periodic waves of the form (Chandrasekhar, 1961)

$$\left.\begin{aligned} \Psi(X,Z,\tau) &= \frac{\sqrt{2}}{\pi^2 a_c}\left(k_2^2 - R_I\right)\Psi_o e^{\sigma\tau}\sin\pi a_c X \sin\pi Z, \\ \Theta(X,Z,\tau) &= \frac{1}{r_E}\frac{4\sqrt{2}\pi}{\left(4\pi^2 - R_I\right)}\Theta_o e^{\sigma\tau}\cos\pi a_c X \sin\pi Z, \end{aligned}\right\} \tag{3.1}$$

where
$$R_{Ec} = \frac{\left(\Lambda k^4 - Da^{-1}k_1^2\right)\left(k_2^2 - R_I\right)\left(4\pi^2 - R_I\right)}{4\pi^4 a^2}, \tag{3.2}$$

$r_E = \frac{R_E}{R_{Ec}}$ and $k^2 = \pi^2\left(1 + a_c^2\right)$, $k_1^2 = \pi^2\left(\varepsilon^{-1} + a_c^2\right)$, $k_2^2 = \pi^2\left(1 + \eta\, a_c^2\right)$.

Eq. (3.2) is the critical value of R_E discussed later in the section. The normal mode solution satisfies the boundary conditions in Eq. (2.11). In Eq. (3.1), πa_c is the horizontal wave number and π is the vertical wavenumber. The quantities ψ_0 and Θ_0 are, respectively, amplitudes of the stream function and temperature. Substituting Eq. (3.1) into the linearized version of Eqs. (2.9) – (2.10) and integrating the above equation with respect to x in $\left[0, \frac{2\pi}{\pi a_c}\right]$, and also with respect to z in [0, 1], we obtain a set of homogeneous equations in Ψ_o and Θ_o. In obtaining a non-trivial solution for ψ_0 and Θ_0, we require

$$r_E = \frac{\left[\sigma + \frac{\Pr}{k^2}\left(\Lambda k^4 - Da^{-1}k_1^2\right)\right]\left(\sigma + k_2^2 - R_I\right)\left(4\pi^2 - R_I\right)}{4\Pr\pi^4 a^2}, \tag{3.3}$$

The onset of convection in Newtonian liquids can occur in one of the following ways:

(i) marginal stationary convection (steady convection),

(ii) marginal oscillatory convection (unsteady convection).

The scaled thermal Rayleigh number r_E is the eigenvalue of the problem that throws light on the stability or otherwise of the system. The critical value of R_E, *i.e.*, R_{Ec} signifies the onset of convection via one of the above modes. It can be proved that the "*Principle of Exchange of Stabilities* (*PES*)" is valid in the problem and hence we consider only the marginal stationary state.

3.1. Marginal stationary state

If σ is real, then the marginal instability occurs when $\sigma = 0$. This gives the stationary thermal Rayleigh number in the form

$$r_E^s = 1. \tag{3.4}$$

The critical wave number a_c satisfies the equation:

$$2\Lambda\eta\pi^4 a_c^6 + \pi^2\left[\Lambda\pi^2(1+2\eta) + \left(\eta Da^{-1} - \Lambda R_I\right)\right] a_c^4 + \left(R_I - \pi^2\right)\left(\Lambda\pi^2 - \frac{Da^{-1}}{\varepsilon}\right) = 0, \qquad (3.5)$$

The linear theory reveals that the stationary convection is the only possible mode of instability and that oscillatory mode can be discounted. The linear theory predicts only the condition for the onset of convection and is silent about the heat transfer. We now embark on a weakly nonlinear analysis by means of a truncated representation of Fourier series for velocity and temperature fields to find the effect of various parameters on finite amplitude convection and to know the amount of heat transfer. Specifically we are considering the most minimal mode for studying nonlinear instability. We note that the results obtained from such an analysis can serve as starting values while solving a more general nonlinear convection problem.

4.Local nonlinear stability theory

The first effect of nonlinearity is to distort the temperature field through the interaction of ψ and Θ . The distortion of temperature field will correspond to a change in the horizontal mean, *i.e.*, a component of the form $\sin(2\pi z)$ will be generated. Thus a minimal double Fourier series which describes the finite amplitude convection in a Newtonian fluid is

$$\left.\begin{aligned} \Psi(X,Z,\tau) &= \frac{\sqrt{2}}{\pi^2 a_c}\left(k_2^2 - R_I\right) A(\tau)\sin\pi a X \sin\pi Z, \\ \Theta(X,Z,\tau) &= \frac{4\pi}{r_E\left(4\pi^2 - R_I\right)}\left[\sqrt{2}B(\tau)\cos\pi a X \sin\pi Z - C(\tau)\sin 2\pi Z\right], \end{aligned}\right\} \qquad (4.1)$$

where the amplitudes *A*, *B* and *C* are to be determined from the dynamics of the system. Substituting Eq. (4.1) into Eqs. (2.9) – (2.10) and following standard orthogonolization procedure for the Galerkin expansion, we obtain the following nonlinear autonomous system (generalized Lorenz model, Sparrow, 1981) of differential equations:

$$\dot{A} = a\,Pr\left(B - A\right), \tag{4.2}$$

$$\dot{B} = k^{-2}\left(1 - R_I k_2^{-2}\right)\left[r_E\,A - B - AC\right], \tag{4.3}$$

$$\dot{C} = k^{-2}\left(1 - R_I k_2^{-2}\right)\left[AB - bC\right], \tag{4.4}$$

where $a = \Lambda - Da^{-1}k_1^2 k^{-4}$, $b = \left(4\pi^2 k_2^{-2} - R_I k_2^{-2}\right)\left(1 - R_I k_2^{-2}\right)^{-1}$ andthe over dot denotes time derivative with respect to τ. It is important to observe that the nonlinearities in the Eqs. (4.2) – (4.4) stem from the convective terms in the energy equation (2.3) as in the classical Lorenz system (Lorenz, 1963).

5.Heat Transport

The horizontally-averaged Nusselt number, Nu, for the stationary mode of convection (the preferred mode in this problem) is given by

$$Nu\left(\tau\right) = \frac{\left[\dfrac{a_c}{2}\displaystyle\int_{X=0}^{2/a_c}\left\{(1-Z)+\Theta\right\}_{,Z}\,dX\right]_{Z=0}}{\left[\dfrac{a_c}{2}\displaystyle\int_{X=0}^{2/a_c}(1-Z)_{,Z}\,dX\right]_{Z=0}}. \tag{5.1}$$

Substituting Eq. (4.1) in Eq. (5.1) and completing the integration, we get

$$Nu\left(\tau\right) = 1 + 2\left[r_E\left(1 - \frac{R_I}{4\pi^2}\right)\right]^{-1} C(\tau). \tag{5.2}$$

The second term on the right side of Eq. (5.2) characterizes the convective contribution to the heat transport. To quantify the Nusselt number we need to know $C(\tau)$ and hence there is a need to solve the Lorenz system (4.2). Alternately we may write $B(\tau)$ and $C(\tau)$ in terms of $A(\tau)$. This is done in the succeeding section.

6.The Ginzburg-Landau equation from the Lorenz model

From the Eqs. (4.2) and (4.3) we have

$$B = \left[\Pr\left(\Lambda - \frac{Da^{-1}k_1^2}{k^4}\right)\right]^{-1} \dot{A} + A \tag{6.1}$$

and

$$C = \frac{1}{A}\left[r_E A - B - \frac{1}{k^{-2}\left(k_2^2 - R_I\right)}\dot{B}\right]. \tag{6.2}$$

Using Eq. (6.1) in Eq. (6.2), we get

$$C = \frac{1}{A}\left[D_1 A - D_2 \dot{A} - D_3 \ddot{A}\right] \tag{6.3}$$

where

$D_1 = r_E - 1$, $D_2 = \left[\Pr\left(\Lambda - \frac{Da^{-1}k_1^2}{k^4}\right)\right]^{-1} + k^2\left(k_2^2 - R_I\right)^{-1}$ and

$$D_3 = k^2\left[\Pr\left(\Lambda - \frac{Da^{-1}k_1^2}{k^4}\right)\left(k_2^2 - R_I\right)\right]^{-1}.$$

Substituting Eqs. (6.1) and (6.3) in Eq. (4.4) we get a third order equation in A and neglecting terms of type $\left(\frac{dA}{d\tau}\right)^2, \frac{dA}{d\tau}\frac{d^2A}{d\tau^2}, A^2\frac{dA}{d\tau}$ and $A\frac{d^2A}{d\tau^2}$, we get the following equation:

$$bD_2 \frac{dA}{d\tau} = bD_1 A - A^3, \tag{6.4}$$

Eq. (6.4) is obviously the Ginzburg-Landau model for non-linear convection in a Newtonian fluid-saturated anisotropic porous medium with heat source. Substituting Eq. (6.4) in Eq. (6.3), we get C in terms of *A* in the form:

$$C = \frac{-D_1^2 D_3}{D_2^2} + \frac{1}{b}\left(1 + \frac{D_1 D_3 \{3b+1\}}{D_2^2}\right) A^2 - \frac{3D_3}{b^2 D_2^2} A^4, \tag{6.5}$$

Solving Eq. (6.4) for $A(\tau),$ we get

$$A(\tau) = \left[\frac{1}{bD_1} + \left(\frac{1}{A(0)^2} - \frac{1}{bD_1}\right) e^{\left\{-\frac{2D_1}{D_2}\tau\right\}}\right]^{-1/2}, \tag{6.6}$$

where $A(0)$ is the initial amplitude. It is apparent from the above that Eq.(5.2) is an analytical expression for the Nusselt number with $C(\tau)$ given by Eq. (6.5).

7.Results and Discussion

Heat transport by thermal convection in porous media with heat source is investigated using a minimal Fourier series. The controlling parameter is the external Rayleigh number, R_E, which is influenced by the heat source (sink) parameter, R_I, (internal Rayleigh number). In view of the fact that we intend studyin Buoyancy induced convection, we assume that R_I is small enough not to induce convection by itself. Hence we have assumed the values of R_I to be in the range *-1 to +1*. Positive values of R_I represent a heat source and negative values of R_I a heat sink. Onset of convection and heat transport are also influenced by inverse Darcy number, Da^{-1}, Brinkman number, Λ and mechanical and thermal anisotropy parameters ε and η. Da^{-1} represents the structure of porous medium, Λ the change in viscosity of the fluid due to the porous medium, ε and η represent the differential packing of the spherical particles in the horizontal and vertical directions. It is the mechanical anisotropy that leads to thermal anisotropy. The porous medium under consideration is a loosely packed one and hence we assume Da^{-1} to take the values from *1 to 1000*. The quantity Λ can assume a range of values that are greater than, equal to or less than *1* (See Givler and Altobelli (1994)), and ε and η are assumed to take values in a neighbourhood involving unity. The values of ε and η greater than *1* would indicate that the horizontal medium or fluid properties are relatively greater than their vertical counterparts.

The minimal representation of Fourier series gives us the Lorenz model for porous media using which we arrived at the Ginzburg-Landau equation of the problem. This equation is a Bernoulli differential equation and has an analytical solution (6.6). Starting from an assumed initial amplitude $A(0)$ we can obtain the amplitude $A(\tau)$ and thereby the other amplitudes $B(\tau)$ and $C(\tau)$. The amplitude $C(\tau)$ is then used to quantify the heat transport in terms of the Nusselt number, *Nu.* The results of extensive computation are shown in the fig. 2.(a), (b),(c) and (d).

From figs. 2.(a), (b),(c) and (d) it is clear that heat source $(R_I > 0)$ enhances the heat transport and heat sink $(R_I < 0)$ diminishes it. It is also seen that heat source delays the heat transport. The effect of inverse Darcy number Da^{-1}, can be seen in the fig. 2. It is observed that in the case of both heat source and heat sink, the effect of Da^{-1} is to augment the heat transport. From fig. 2(a) it is clear that Da^{-1} enhances the heat transport in the range $Da^{-1} = 1 \text{ to } 100$. But as Da^{-1} increases further (i.e., $Da^{-1} = 1000$), heat transfer diminishes initially and after a short time it increases. Fig.2(b) is a plot of *Nu* versus τ for different values of Λ. The effect of Λ is to decrease the heat transport in the case of both heat source and heat sink. The effects of ε and η on *Nu* is opposite to each other which is seen in the figs.2(c) and (d). In the case of both, heat source and heat sink, ε decreases the heat transport and η increases the same.

References

Ali Nouri-Borujerdi, A ,Noghrehabadi, A. R., Rees, D. A. S.(2008). Influence of Darcy number on the onset of convection in a porous layer with a uniform heat source, *Int. J. Ther. Sci.*, 47, 1020-1025.

Crolet, J. M. (2000). Computation methods for flow and transport in porous media, *Kluwer Academic Press*.

Givler, R. C. and Altobelli, A. (1994). A determination of the effective viscosity for Brinkman-Forchheimer model. *J. Fluid Mech.*, 258, 355-370.

Ingham, D. B. and Pop, I. (1998). Transport phenomena in porous media, *Elsevier*.

Israel-Cookey, C., Omubo-Pepple, V. B., B. Obi, B. I. and Eze, L. C.(2010). Onset of thermal instability in a low Prandtl number fluid with internal heat source in a porous medium, *Am. J. Sci. Ind. Res*.,1(1), 18-24.

Kaviany, M. (1995): Principles of heat transfer in porous media, 2ndEdn. *Springer-Verlag*.

Khalili A. and Shivakumara I. S. (1998). Onset of convection in a porous layer with net through-flow and internal heat generation, *Phys. of fluids*, 10, 315.

Khalili, A., Shivakumara, I. S. and Huettel, M. (2002). Effects of through-flow and internal heat generation on convective instabilities in an anisotropic porous layer. *J. Porous Media*, 5,181-198.

Nield, D. A. and Bejan, A. (2006). Convection in porous media. Second ed., *Springer-Verlag*, New York.

Riahi, N.(1990). Non-linear convection in a porous medium with internal heat source, *Int. J. Engg. Sci.*, 28(6), 497-503.

Rees, D. A. S. and Pop, I. (1995). Free convection induced by a vertical wavy surface with uniform heat flux in a porous medium ,*Trans. A.S.M.E. Journal of Heat Transfer*. 117, 547-550.

Somerton, C. W., McDonough, J. M. and Catton, I.(1984). Natural convection in a volumetrically heated porous layer, *ASME J. Heat Transfer*, 106(1), 241-244.

Vadasz, P. (1998). Free convection in a porous media, *Transport Phenomena in Porous Media*,*Elsevier*(Ed. By Ingham, D. B. and Pop, I.).

Vafai, K. (2000). Hand book of porous media, *Marcel Dekker*, Inc., New York.

Veronis, G. (1966). Motions at subcritical values of the Rayleigh number in a rotating fluid. *J. Fluid Mech.*, 24, 545–554.

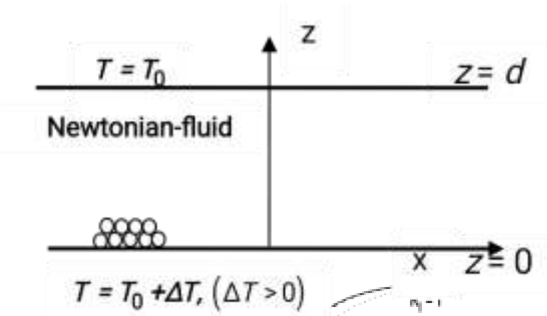
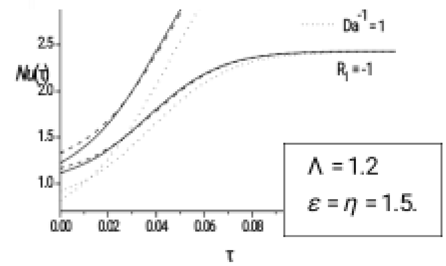
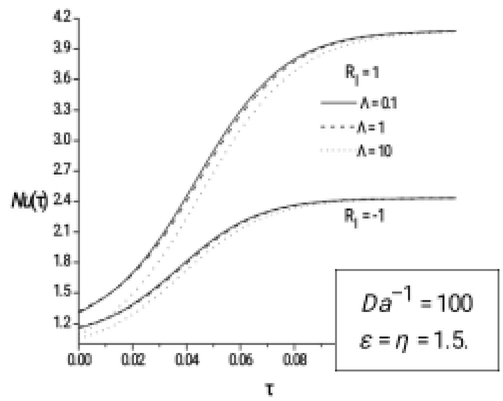
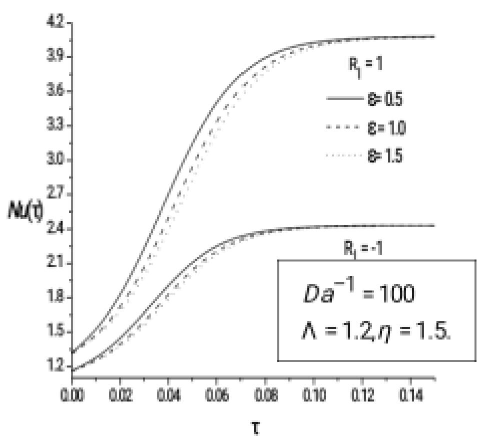


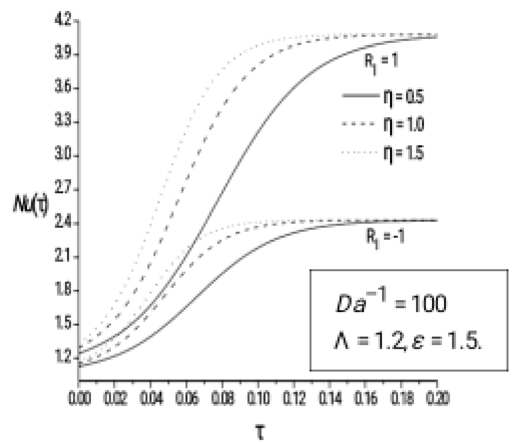
Fig. 1: Physical configuration.





(a) (b)





(c) (d)

Fig. 2. Plot of Nusselt number *Nu* versus time τ for different values of (a) inverse Darcy number Da^{-1},

(b) Brinkman number Λ, (c) mechanical anisotropy parameter ε and (d) thermal anisotropy parameter η.