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Generalized Banach Space Valued Difference Sequence Spaces and Their Köthe-Toeplitz Duals

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Abstract

In this paper, we develop $c_0(\Delta^m, X), c(\Delta^m, X)$ and $\ell_\infty(\Delta^m, X)$ respectively and determine their Köthe-Toeplitz duals in operator form.

1 Introduction

The scalar sequence spaces $c_0(\Delta^m), c(\Delta^m)$ and $\ell_\infty(\Delta^m)$, where for instance

$$\ell_\infty(\Delta^m) = \{x = (x_k): \Delta^m x_k \in \ell_\infty\}$$

were defined and studied by Et and Colak [2], which are in fact the generalization of some results of Kizmaz [3] and Et [1].

Let U denote the set of all sequences

$$u = (u_k)_{k=1}^{\infty}$$

such that $(u_k) \neq 0 (k = 1,2,3, \dots)$. Given any $u \in U$, we write

$$\frac{1}{u} = \left(\frac{1}{u_k}\right)_{k=1}^{\infty} \quad \text{and} \quad p = (p_k)_{k=1}^{\infty}$$

shall always be an arbitrary sequence of positive reals.
We define the Δ^m difference sequence spaces of Banach space X-valued sequences:

$$c_0(\Delta^m, X) = \{\bar{x} = (x_k): x_k \in X \text{ and } \Delta^m x \in c_0(X)\} \tag{1.1}$$

$$c(\Delta^m, X) = \{\bar{x} = (x_k): x_k \in X \text{ and there exists } \ell \in X \text{ such that } (\Delta^m x - \ell) \in c_0(X)\}$$

$$\ell_\infty(\Delta^m, X) = \{\bar{x} = (x_k): x_k \in X \text{ and } \Delta^m x \in \ell_\infty(X)\}$$

where $m \in \mathbb{N}$,

$$\Delta^0 x = (x_k), \Delta x = (x_k - x_{k+1})$$
$$\Delta^m x = (\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1})$$

and so that

$$\Delta^m x_k = \sum_{n=0}^{m} (-1)^n \binom{m}{n} x_{k+n}$$

where

$$\binom{m}{n} = {}^{m}C_n$$

These spaces are Banach spaces with the norm defined by

$$\|\bar{x}\|_m = \sum_{i=1}^{m} \|x_i\| + \|\Delta^m x\|_\infty \tag{1.4}$$

Let us define the operator

$$S_m : \ell_\infty(\Delta^m, X) \to \ell_\infty(\Delta^m, X)$$

defined by

$$S_m \bar{x} = (0,0, \dots, x_{m+1}, x_{m+2}, \dots)$$

where

$$\bar{x} = (x_1, x_2, x_3, \dots)$$

is a bounded linear operator on $\ell_\infty(\Delta^m, X)$.
Furthermore, the set

$$S_m[\ell_\infty(\Delta^m, X)] = S_m \ell_\infty(\Delta^m, X)$$
$$= \{\bar{x} = (x_k) : x_k \in \ell_\infty(\Delta^m, X), x_1 = x_2 = x_3 = \dots = x_m = 0\}$$

is a subspace of $\ell_\infty(\Delta^m, X)$ and

$$\|\bar{x}\|_m = \|\Delta^m x\|_\infty \text{ in } S_m \ell_\infty(\Delta^m, X)$$

Definition:

Let X and Y be Banach spaces and (A_k) a sequence of linear but not necessarily bounded operators A_k on X into Y.

Suppose $E(X)$ is a non-empty set of X-valued sequences. The α-dual of $E(X)$ is defined by

$$E^\alpha(X) = \left\{ \bar{A} = (A_k) : \sum_{k=1}^{\infty} \|A_k x_k\| \text{ converges for all } (x_k) \in E(X) \right\}.$$

A decisive break with the classical approach was made by Robinson [5] in 1950, when he considered the action of infinite matrices of linear operators from a Banach space of sequences of elements of that space. The KötheToeplitz duals for various vector valued sequence spaces have been obtained in terms of sequences of operators by Maddox [4].

Theorem 1.1: The sequence space $l_\infty(\Delta^m, X)$ is a Banach space with the norm defined in (1.4).

Proof: Let $(\bar{x}^r)$ be a Cauchy sequence in $l_\infty(\Delta^m, X)$ where

$$(\bar{x}^r) = (x_i^r) = (x_1^r, x_2^r, x_3^r, \dots) \in l_\infty(\Delta^m, X), \forall r \in \mathbb{N}.$$

Then

$$\|\bar{x}^r - \bar{x}^s\|_m = \sum_{i=1}^{m} \|x_i^r - x_i^s\| + \sup_{k \geq m} \|\Delta^m(x_k^r - x_k^s)\| \to 0 \ \text{ as } r, s \to \infty.$$

Hence we get

$$\|x_k^r - x_k^s\| \to 0 \ \text{ as } r, s \to \infty, \ \text{ for each } k \in \mathbb{N}.$$

Therefore (x_k^r) is a Cauchy sequence in X. Since X is complete, it is convergent, i.e.

$$\lim_{r \to \infty} x_k^r = x_k \text{ (say), for each } k \in \mathbb{N}.$$

Since ($\bar{x}^r$) is a Cauchy sequence, for each $\varepsilon > 0$ there exists $N_0 = N_0(\varepsilon)$ such that

$$\|\bar{x}^r - \bar{x}^s\|_m < \varepsilon, \forall r, s \geq N_0$$

Hence

$$\sum_{i=1}^{m} \|x_i^r - x_i^s\| \leq \varepsilon$$

and

$$\left\| \sum_{n=0}^{m} (-1)^n \binom{m}{n} (x_{k+n}^r - x_{k+n}^s) \right\| \leq \varepsilon, \forall r, s \geq N_0, k \in \mathbb{N}$$

So we have

$$\lim_{s \to \infty} \sum_{i=1}^{m} \|x_i^r - x_i^s\| = \sum_{i=1}^{m} \|x_i^r - x_i\| \leq \varepsilon.$$

and

$$\lim_{s \to \infty} \|\Delta^m(x_k^r - x_k^s)\| = \|\Delta^m(x_k^r - x_k)\| < \varepsilon, \forall r \geq N_0$$

This implies that

$$\|\bar{x}^r - \bar{x}\|_m \leq 2\varepsilon, \forall r \geq N_0$$

i.e.

$$\bar{x}^r \to \bar{x} \text{ as } r \to \infty$$

Now

$$\|\Delta^m x_k\| = \left\|\sum_{n=0}^{m} (-1)^n \binom{m}{n} x_{k+n}\right\|$$

$$= \left\|\sum_{n=0}^{m} (-1)^n \binom{m}{n} \left(x_{k+n} - x_{k+n}^N + x_{k+n}^N\right)\right\|$$

and

$$\lim_{s\to\infty} \|\Delta^m (x_k^r - x_k^s)\| = \|\Delta^m (x_k^r - x_k)\| < \varepsilon, \forall r \geq N_0$$

This implies that

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$$\|\Delta^m x_k\| = \left\|\sum_{n=0}^{m} (-1)^n \binom{m}{n} x_{k+n}\right\|$$

$$= \left\|\sum_{n=0}^{m} (-1)^n \binom{m}{n} \left(x_{k+n} - x_{k+n}^N + x_{k+n}^N\right)\right\|$$

$$\leq \left\|\sum_{n=0}^{m} (-1)^n \binom{m}{n} \left(x_{k+n}^N - x_{k+n}\right)\right\| + \left\|\sum_{n=0}^{m} (-1)^n \binom{m}{n} x_{k+n}^N\right\|$$

$$\leq \|\bar{x}^N - \bar{x}\|_m + \left\|\Delta^m x_k^N\right\| = O(1)$$

This implies that

$$\bar{x} \in \ell_\infty(\Delta^m, X)$$

Therefore
$\ell_\infty(\Delta^m, X)$ is a Banach space. This completes the proof.

Lemma 1.1

$$\sup_k \|\Delta^{m-1}x_k - \Delta^{m-1}x_{k+1}\| < \infty \text{ iff}$$

(i)

$$\sup_k k^{-1}\|\Delta^{m-1}x_k\| < \infty$$

(ii)

$$\sup_k \|\Delta^{m-1}x_k - k(k+1)^{-1}\Delta^{m-1}x_{k+1}\| < \infty$$

Proof:

Putting

$$\Delta^{m-1}x_k \text{ instead of } x_k$$

in the proof of lemma

$$\sup_{k\geq 1} \|x_k - x_{k+1}\| < \infty \text{ iff}$$

(i)

$$\sup_{k\geq 1} k^{-1}\|x_k\| < \infty$$

(ii)

$$\sup_{k\geq 1} \|x_k - k(k+1)^{-1}x_{k+1}\| < \infty$$

we get immediately the above result.

Lemma 1.2.

$\sup_k k^{-i}\|\Delta x_k\| < \infty$ implies that $\sup_k k^{-(i+1)}\|x_k\| < \infty$, for all $i \in \mathbb{N}$.
Proof. Given

$$\sup_k k^{-i}\|\Delta x_k\| < \infty$$

then

$$\|x_1 - x_{k+1}\| = \left\| \sum_{l=1}^{k} (x_l - x_{l+1}) \right\| \\ \leq \sum_{l=1}^{k} \|\Delta x_l\| \\ = O\left(k^{i+1}\right)$$

Now

$$\|x_k\| = \|x_k - x_{k+1} + x_{k+1} - x_1 + x_1\| \\ \leq \|x_k - x_{k+1}\| + \|x_1 - x_{k+1}\| + \|x_1\|$$

Which gives

$$k^{-(i+1)}\|x_k\| \leq k^{-(i+1)}\|\Delta x_k\| + k^{-(i+1)}\|x_1 - x_{k+1}\| + k^{-(i+1)}\|x_1\| \\ = O(1)$$

Hence

$$\sup_k k^{-(i+1)}\|x_k\| < \infty$$

This completes the proof.

Lemma 1.3.

$\sup_k k^{-i}\|\Delta^{m-1}x_k\| < \infty$ implies that $\sup_k k^{-(i+1)}\left\|\Delta^{m-(i+1)}x_k\right\| < \infty, \forall i, m \in \mathbb{N}, 1 \leq i < m$
Proof. Putting $\Delta^{m-1}x_k$ instead of Δx_k in Lemma 1.2, we get immediately the above result.

Corollary 1.1.

$$\sup_k k^{-1}\|\Delta^m x_k\| < \infty \text{ implies } \sup_k k^{-m}\|x_k\| < \infty.$$

Corollary 1.2.

$$\bar{x} \in \ell_\infty(\Delta^m, X) \text{ implies that } \sup_k k^{-m}\|x_k\| < \infty.$$

Theorem 1.2.

The α-dual of $S_m c_0(\Delta^m, X), S_m c(\Delta^m, X)$ and $S_m \ell_\infty(\Delta^m, X)$ are

$$S_m c_0^\alpha(\Delta^m, X) = S_m c^\alpha(\Delta^m, X) = S_m \ell_\infty^\alpha(\Delta^m, X) = M_m(B(X, Y))$$

where

$$M_m(B(X,Y)) = \left\{\bar{A} = (A_k): A_k \in B(X,Y), \sum_{k=1}^{\infty} k^m \|A_k\| < \infty\right\} \tag{1.5}$$

Proof: Since

$$S_m c_0(\Delta^m, X) \subset S_m c(\Delta^m, X) \subset S_m \ell_\infty(\Delta^m, X)$$

implies that

$$S_m \ell_\infty^\alpha(\Delta^m, X) \subset S_m c^\alpha(\Delta^m, X) \subset S_m c_0^\alpha(\Delta^m, X)$$

Therefore we have
(i) $M_m(B(X,Y)) \subset S_m \ell_\infty^\alpha(\Delta^m, X)$ and
(ii) $S_m c_0^\alpha(\Delta^m, X) \subset M_m(B(X,Y))$.
(i) Let $\bar{A} = (A_k) \in M_m(B(X,Y))$ and $\bar{x} = (x_k) \in S_m \ell_\infty(\Delta^m, X)$.

Then

$$\sum_{k=1}^{\infty} \|A_k x_k\| \le \sum_{k=1}^{\infty} k^m \|A_k\| \frac{\|x_k\|}{k^m}$$
$$\le \sup_{k\ge 1} \frac{\|x_k\|}{k^m} \sum_{k=1}^{\infty} k^m \|A_k\| < \infty$$

[using corollary (1.2) & equation (1.5)]
Hence $\bar{A} = (A_k) \in S_m \ell_\infty^\alpha(\Delta^m, X)$.
(ii) Let us suppose that $(A_k) \in S_m c_0^\alpha(\Delta^m, X)$, but $A_k \notin M_m(B(X,Y))$. Then there exists an increasing sequence ($x(i)$) such that

$$\sum_{k=n(i)+1}^{n(i+1)} k^m \|A_k\| > i, \forall i \in \mathbb{N}$$

Let $z \in X$ with $\|z\| = 1$.
Define a sequence $\bar{x} = (x_k)$

$$x_k = \frac{k^m z}{i}, n(i) < k \le n(i+1)$$

$z \in X$ is so chosen as

$$\sup_{\|z\|=1} \sum_{k=n(i)+1}^{n(i+1)} \|A_k x_k\| > 1$$

The choice of such a $z \in X$ is possible.

$$\sup_{\|z\|=1} \sum_{n(i)+1}^{n(i+1)} \|A_k x_k\| = \sup_{\|z\|=1} \sum_{n(i)+1}^{n(i+1)} \frac{k^m}{i} \|A_k z\|$$

$$= \sum_{n(i)+1}^{n(i+1)} \frac{k^m}{i} \|A_k\| > 1$$

Then

$$\|\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1}\| \leq \frac{m}{i} \to 0 \text{ as } i \to \infty$$

Thus $(x_k) \in s_m c_0(\Delta^m, X)$ but

$$\sum_{n(i)+1}^{n(i+1)} \|A_k x_k\| > 1$$

Therefore

$$\sum_{n(i)+1}^{n(i+1)} \|A_k x_k\|$$

diverges, which is a contradiction to the fact

$$\sum_{k=1}^{\infty} \|A_k x_k\| < \infty$$

This completes the proof.

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