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International Journal of Research in Engineering and Applied Sciences(IJREAS)
Available online at http://euroasiapub.org/journals.php
Vol. 13 Issue 11, November -2023,
ISSN(O): 2249-3905, ISSN(P) : 2349-6525 | Impact Factor: 8.202| Thomson Reuters ID: L-5236-2015

Generalizations of Certain Special Functions and Hypergeometric Functions

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Abstract

The study of special functions plays a crucial role in mathematics and its applications in various fields. This abstract introduces the concept of generalizations of special functions, particularly hypergeometric functions. Hypergeometric functions are powerful tools that arise in diverse mathematical contexts, from calculus to number theory and physics. In this research, we explore broader classes of functions that encompass and extend the properties of traditional hypergeometric functions. These generalizations aim to provide a more comprehensive framework for solving complex mathematical problems and modeling real-world phenomena. By examining the underlying structures and properties of these extended special functions, we can uncover new connections, uncover hidden symmetries, and develop innovative techniques for solving differential equations and integral transforms. This research has the potential to advance our understanding of mathematical analysis and its applications in various scientific disciplines, making it a significant area of study in contemporary mathematics.

Keywords:-Special Functions, Hypergeometric Functions, Generalizations

Introduction

Special functions have a long and illustrious history in mathematics, arising in a multitude of mathematical contexts and scientific applications. Among these special functions, hypergeometric functions stand out as some of the most versatile and powerful tools in mathematical analysis. However, as mathematical research advances and the need for more flexible and comprehensive solutions arises, the study of generalizations of special functions becomes increasingly important.This research aims to explore and develop generalizations of certain special functions, with a primary focus on extending the theory and applications of hypergeometric functions. Hypergeometric functions, denoted by the symbol ${}_pF_q$, are a class of functions that encompass a wide range of mathematical functions, including binomial coefficients, polynomials, and various transcendental functions. They play a fundamental role in solving differential equations, integral transforms, and generating functions, making them essential in diverse fields such as physics, engineering, statistics, and number theory.

The motivation behind this research lies in the quest for a more comprehensive framework that unifies and extends the properties of traditional hypergeometric functions. By generalizing these special functions, we can uncover hidden connections between seemingly unrelated mathematical concepts and explore new avenues for solving complex problems. These generalizations often involve relaxing some of the constraints of the original hypergeometric functions, such as allowing multivariate or matrix-valued arguments, or introducing additional parameters.One key aspect of this research is the investigation of mathematical structures and symmetries within the generalized special functions. These structures provide insight into their properties and behaviors, leading to a deeper understanding of their mathematical nature. Additionally, the study of special function generalizations has direct implications for real-world applications, as these functions frequently arise in physical and engineering problems, particularly in quantum mechanics, statistical physics, and electromagnetic theory.this research delves into the world of generalizations of certain special functions, primarily focusing on the extensions of hypergeometric functions. By exploring the underlying structures, mathematical properties, and applications of these generalizations, we hope to contribute to the advancement of mathematical analysis and its practical utility in various scientific disciplines. This work stands at the intersection of pure mathematics and applied science, highlighting the enduring significance of special functions in contemporary mathematics.

Need of the Study

The study of generalizations of certain special functions, particularly hypergeometric functions, is driven by a compelling need to expand the mathematical toolkit and address the

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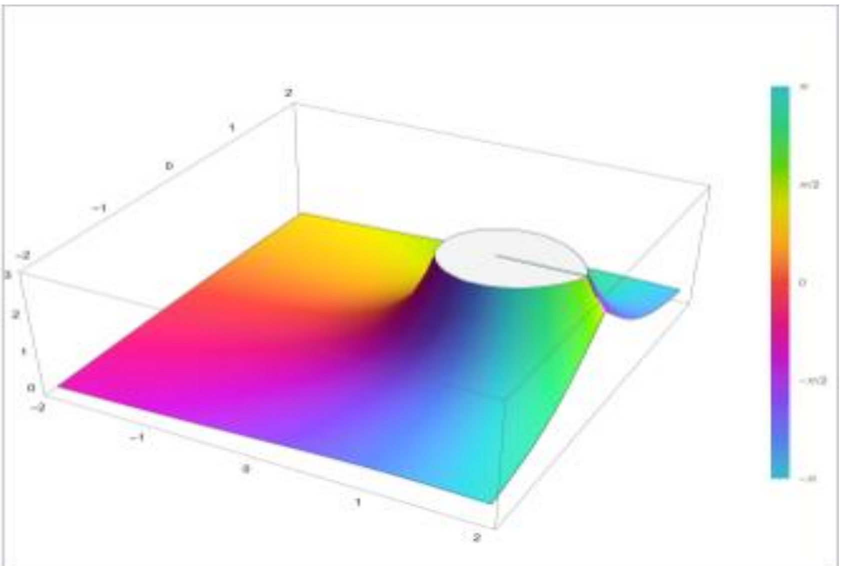
complexities of modern scientific and engineering problems. Special functions have long been essential in mathematics, offering solutions to a wide range of mathematical equations. However, as research across various disciplines becomes more intricate and multidimensional, the limitations of traditional special functions become apparent. Generalizations of these functions enable researchers to adapt to these challenges, providing the flexibility to handle multivariate data, uncover hidden mathematical structures, and find innovative solutions. This research not only contributes to the advancement of mathematical theory but also has a profound impact on practical applications, where these functions are indispensable for modeling and solving complex real-world problems. In essence, the study of generalizations of special functions stands at the forefront of mathematical innovation, bridging the gap between theory and practicality in diverse scientific fields.

Hypergeometric Function

In mathematics, a generalized hypergeometric series is a series where the ratio between consecutive coefficients, indexed by n, follows a rational function of n. When this series converges, it defines a generalized hypergeometric function. This function can be extended over a broader domain of its argument through analytic continuation.



While the term "hypergeometric series" is often used to refer to these generalized series, it can

Vol. 13 Issue 11, November -2023,
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sometimes specifically indicate the Gaussian hypergeometric series. Generalized hypergeometric functions encompass a wide range of special cases, including the Gaussian hypergeometric function and the confluent hypergeometric function. These, in turn, give rise to various specific special functions for particular cases, such as elementary functions, Bessel functions, and classical orthogonal polynomials.

The generalized hypergeometric function is defined and represented as follows:

$$ {}_2F_1(\xi,b;c;z) = \sum_{n=0}^{\infty} \frac{(\xi)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \qquad c \neq 0,-1,-2,\ldots $$

$$ \text{where}(\xi)_n = \xi(\xi+1)\ldots(\xi+n-1); (\xi)_0 = 1. $$

The series ${}_2F_1(\xi ,b; c;z)$ is convergent in the following cases;

Where $|z| < 1$, the series is convergent.

When $z = 1$, the series is absolutely convergent for $\Re(c-\xi-b) > 0$.

When $z = -1$, the series is absolutely convergent for $\Re(c-\xi-b) > -1$.

A special case of the series is ${}_1F_1(b; c;z)$ called the confluent hypergeometric function, which is defined as

$$ {}_1F_1(b;c;z) = \sum_{n=0}^{\infty} \frac{(b)_n}{(c)_n} \frac{z^n}{n!}, $$

the series in is absolutely convergent for all values of b, c and z,

$c \neq 0,-1,-2,\ldots$

$$ {}_pF_q \left[\begin{matrix} \zeta_i \ldots \zeta_p \\ \eta_1 \ldots \eta_q \end{matrix} ;z \right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} (\zeta_i)_n}{\prod_{j=1}^{q} (\eta_j)_n} \frac{z^n}{n!} \quad \eta_j \neq 0,-1,-2; (j=1,\ldots.q). $$

The series in is convergent for all values of z, when $p \leq q$; when $p = q+1$, the series is convergent for |z| <1 ; when z = 1, the series converges if

$$\Re\left[\sum_{j=1}^{q}(\eta_j)-\sum_{i=1}^{p}(\zeta_i)\right]>0$$

and when z = −1, it converges if

$$\Re\left[\sum_{j=1}^{q}(\eta_j)-\sum_{i=1}^{p}(\zeta_i)\right]>-1.$$

Differentiation formulas

Using the identity $(a)_{n+1}=a(a+1)_n$ it is shown that

$$\frac{d}{dz}\,{}_2F_1(a,b;c;z)=\frac{ab}{c}\,{}_2F_1(a+1,b+1;c+1;z)$$

and more generally,

$$\frac{d^n}{dz^n}\,{}_2F_1(a,b;c;z)=\frac{(a)_n(b)_n}{(c)_n}\,{}_2F_1(a+n,b+n;c+n;z)$$

Literature Review

Virchenko, N.,et al (2001). The generalized hypergeometric function, denoted as pF_q, is a powerful mathematical tool used to express complex mathematical functions in a concise and elegant manner.The generalized hypergeometric function exhibits a wide range of applications in mathematics, physics, and engineering. This function is integral to special function theory, where it plays a fundamental role in expressing solutions to various differential equations, including Bessel's equation and Legendre's equation. It serves as a bridge between different types of special functions, allowing for the transformation and simplification of complex mathematical expressions. The generalized hypergeometric function appears in various areas, such as quantum mechanics, electromagnetism, and statistical mechanics.

BeukersF.(2014). Hypergeometric functions hold a unique and esteemed position in the realm of mathematical functions. Their exceptional nature stems from their remarkable versatility and widespread utility. These functions, denoted as pF_q, where p and q are non-negative integers, are celebrated for their ability to solve a wide array of complex mathematical problems.Their specialness is evident in their capacity to unify a broad spectrum of special functions, including polynomial, exponential, trigonometric, and logarithmic functions. Hypergeometric functions act as a connective thread between these disparate mathematical tools, allowing for the simplification of intricate expressions. Their convergence properties and well-defined series representations make them amenable to analytical study and efficient numerical evaluation.

Srivastava, H. M., Agarwal, P., & Jain, S. (2014). Generating functions are powerful mathematical tools that provide elegant ways to represent and manipulate mathematical functions. For generalized Gauss hypergeometric functions, the use of generating functions offers a systematic approach to understand and derive these functions. These functions, often denoted as F(a, b; c; z), are solutions to the Gauss hypergeometric differential equation, and their generating functions play a pivotal role in their study. This generating function elegantly connects the hypergeometric function to a power series, allowing for its expansion and manipulation.These generating functions are instrumental in solving differential equations and evaluating definite integrals involving generalized Gauss hypergeometric functions. They also enable the derivation of various properties and relationships among different hypergeometric functions.

Rahman, G., et al (2020). The development of new generalizations of extended beta and hypergeometric functions represents a significant stride in mathematical research, offering fresh insights into the interplay between various special functions. These generalized functions are designed to encompass and extend the utility of traditional beta and hypergeometric functions, making them even more versatile and applicable in diverse contexts. These additional parameters expand the range of possible applications and allow for a broader class of integrals and series to be represented. These new generalizations find applications in various mathematical and scientific fields, including combinatorics, number theory, probability theory, and physics. They enable mathematicians and scientists to tackle a broader set of problems, providing tools to represent and manipulate functions that were previously challenging to work with. In summary, the development of these new generalizations enhances the mathematical toolkit, broadening the scope of problems that can be addressed and underscoring their significance in both theoretical and practical contexts.

Agarwal, P., et al (2015). Extended hypergeometric functions of two and three variables are advanced mathematical constructs that serve as natural extensions of the classical hypergeometric functions. These functions provide a powerful framework for expressing and solving a wide range of mathematical and physical problems, offering more flexibility and broader applicability.In the case of extended hypergeometric functions with two variables, denoted as F(a, b; c; x, y), these functions can be expressed as double series or as integrals, allowing for the manipulation and analysis of complex mathematical expressions. They are essential in solving partial differential equations and finding solutions to problems in mathematical physics, particularly in the context of conformal mapping, potential theory, and fluid dynamics. The generalization to two variables enhances their ability to describe phenomena in a wider range of physical systems and mathematical models.Extending these functions to three variables, F(a, b, c; x, y, z), further increases their versatility. They find applications in advanced topics such as hypergeometry and modular forms.

Yang, X. J. (2021). The theory and applications of special functions hold immense significance for scientists and engineers across various fields, providing them with powerful tools to solve complex problems and model real-world phenomena. Special functions encompass a wide range of mathematical functions that are not only distinctive but also possess specific properties that make them indispensable in both theoretical and practical contexts.These functions include, but are not limited to, Bessel functions, Legendre polynomials, hypergeometric functions, and more. They find extensive applications in physics, engineering, and various branches of mathematics. For scientists and engineers, these functions serve as fundamental building blocks in solving differential equations, modeling wave phenomena, and understanding physical systems.

Srivastava, R. (2014). Generating functions play a pivotal role in mathematics and its applications, and they are particularly significant in connection with a certain family of extended and generalized hypergeometric functions. These generating functions serve as bridges between the worlds of combinatorics, calculus, and special functions, offering a structured way to express and manipulate complex mathematical expressions.One class of generating functions associated with these hypergeometric functions is the Mellin-Barnes representation. These generating functions express hypergeometric functions as integrals over the complex plane, providing a powerful method for evaluating them numerically and analytically.

Srivastava, R. (2013). Generalizations of Pochhammer's symbol, often referred to as Pochhammer's extended symbols, have opened up exciting avenues in mathematics, leading to the development of associated families of hypergeometric functions and hypergeometric polynomials. These extensions allow for a broader range of applications and provide versatile tools for solving mathematical problems across various disciplines. These q-hypergeometric functions are powerful tools for representing and solving problems involving series and integrals.Pochhammer's symbols have been extended to include generalized hypergeometric functions like Lauricella and Kampé de Fériet functions. These multi-parameter functions offer increased flexibility and utility in solving partial differential equations and other complex mathematical problems, particularly in the field of hypergeometry. Associated hypergeometric polynomials, such as the Jacobi, Legendre, and Meixner polynomials, arise from these generalizations.

The gamma function

The upper incomplete gamma function, denoted as Γ(a, x), is a mathematical function that is a part of the gamma function and is defined as follows:

$$\Gamma(s,x) = \int_x^\infty t^{s-1} e^{-t}\, dt,$$

and the lower incomplete gamma function is defined as

$$\gamma(s,x) = \int_0^x t^{s-1} e^{-t} dt.$$

The functional behaviour of these functions and the decomposition formula

$$\Gamma(s) = \Gamma(s,x) + \gamma(s,x).$$

Extensions of the incomplete gamma function

$$\gamma(s,x;b) = \int_0^x t^{s-1} \exp\left(-t - \frac{b}{t}\right) dt$$

And

$$\Gamma(s,x;b) = \int_x^\infty t^{s-1} \exp\left(-t - \frac{b}{t}\right) dt.$$

Beta Function and Its Extensions

The (Euler's) classical beta function B(α,β) which is defined by

$$B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} dt,$$

where $\Re(\alpha) > 0, \Re(\beta) > 0.$

Introduced the first extensions of the classical beta function as follows

$$B_p(\xi,\zeta) = B(\xi,\zeta;p) = \int_0^1 t^{\xi-1}(1-t)^{\zeta-1} \exp\left(-\frac{p}{t(1-t)}\right) dt,$$
$$(\Re(p) \geq 0).$$

Introduced the further extend of the beta function as follows:

$$B_p^{(\mu,\nu)}(\xi,\zeta) = \int_0^1 t^{\xi-1}(1-t)^{\zeta-1}{}_1F_1\left(\mu;\nu;-\frac{p}{t(1-t)}\right)dt,$$

where $\Re(p) \geq 0; \min[\Re(\xi),\Re(\zeta),\Re(\mu),\Re(\nu)] > 0.$

Results of New Extended Gauss and Confluent Hypergeometric Function

Here by presenting the numerical values of MC.F_m(η,ζ ;ξ ;z) and MC.Φm(ζ ;ξ ;z) in Table 3.3 and Table for m = −2 : 1 : 2. When m = 0 then get the values of Gauss hypergeometric function and confluent hypergeometric function.

Table 1 Numerical values of MC.F_m(2,1;3;z)

z	$m=-2$	$m=-1$	$m=0$	$m=1$	$m=2$
0.0000	0.7248	0.8489	1.0000	1.1846	1.4107
0.1000	0.7734	0.9080	1.0721	1.2728	1.5188
0.2000	0.8308	0.9778	1.1572	1.3768	1.6463
0.3000	0.9000	1.0618	1.2594	1.5017	1.7993
0.4000	0.9855	1.1653	1.3853	1.6552	1.9870
0.5000	1.0947	1.2972	1.5452	1.8495	2.2240
0.6000	1.2410	1.4730	1.7572	2.1061	2.5356
0.7000	1.4512	1.7236	2.0570	2.4664	2.9701
0.8000	1.7914	2.1235	2.5295	3.0270	3.6382
0.9000	2.4976	2.9335	3.4632	4.1089	4.8982
1.0000	6.7829	7.5302	8.4142	9.4648	10.7195

Table 2 Numerical values of MC. $_1F_1$,m(1;3;z)

z	$m=-2$	$m=-1$	$m=0$	$m=1$	$m=2$
0.0000	0.7248	0.8489	1.0000	1.1846	1.4107
0.1000	0.7478	0.8769	1.0342	1.2264	1.4620
0.2000	0.7721	0.9064	1.0701	1.2704	1.5159
0.3000	0.7976	0.9374	1.1080	1.3167	1.5726
0.4000	0.8245	0.9701	1.1478	1.3654	1.6324
0.5000	0.8528	1.0045	1.1898	1.4167	1.6953
0.6000	0.8826	1.0408	1.2340	1.4707	1.7616
0.7000	0.9141	1.0790	1.2806	1.5277	1.8314
0.8000	0.9473	1.1194	1.3298	1.5879	1.9051
0.9000	0.9824	1.1620	1.3817	1.6513	1.9828
1.0000	1.0195	1.2070	1.4366	1.7183	2.0649

Generalized functions and their properties

Generalized functions, also known as distributions, are mathematical objects used to extend the concept of functions to include more irregular or singular cases. They play a fundamental role in the theory of partial differential equations and provide a framework for solving equations involving singularities or distributions of mass.

$$^{\Psi}\hat{\Gamma}_p(x) := {}^{\Psi}\Gamma_p\left[\begin{matrix}(\beta_i,\alpha_i)_{1,\xi}\\(\mu_j,\kappa_j)_{1,\eta}\end{matrix}\middle| x\right] = \int_0^{\infty} t^{x-1}{}_{\xi}\Psi_{\eta}\left(-t-\frac{p}{t}\right)dt$$

Key properties of generalized functions include linearity, which allows for the superposition of distributions, and the notion of distributional derivatives, which extends differentiation to distributions. For example, the Dirac delta function, δ(x), is a well-known generalized function that is zero everywhere except at x = 0, where it "spikes" to infinity, satisfying δ'(x) = -δ(x)'.

$$^{\Psi}\hat{B}_p(x,y) := {}^{\Psi}B_p\left[\begin{matrix}(\beta_i,\alpha_i)_{1,\xi}\\(\mu_j,\kappa_j)_{1,\eta}\end{matrix}\middle| x,y\right] = \int_0^{1} t^{x-1}(1-t)^{y-1}{}_{\xi}\Psi_{\eta}\left(-\frac{p}{t(1-t)}\right)dt.$$

Another essential property is the action of a distribution on a test function, resulting in a real number. This pairing between distributions and test functions is known as duality, facilitating the integration of distributions with regular functions.Generalized functions provide a powerful mathematical tool for solving a wide range of problems in physics, engineering, and mathematics, where conventional functions may fail to describe complex phenomena or singularities accurately.

Theorem 1. The following equality holds true:

$$ {}^{\Psi}\hat{\Gamma}_p(x){}^{\Psi}\hat{\Gamma}_p(y) = 4\int_0^{\frac{\pi}{2}}\int_0^{\infty} r^{2(x+y)-1}(\cos\theta)^{2x-1}(\sin\theta)^{2y-1} \times {}_{\xi}\Psi_{\eta}\left(-r^2(\cos\theta)^2-\frac{p}{r^2(\cos\theta)^2}\right) \times {}_{\xi}\Psi_{\eta}\left(-r^2(\sin\theta)^2-\frac{p}{r^2(\sin\theta)^2}\right)drd\theta. $$

Substituting t = u²in , we get

$$ {}^{\Psi}\hat{\Gamma}_p(x) = 2\int_0^{\infty} u^{2x-1}{}_{\xi}\Psi_{\eta}\left(-u^2-\frac{p}{u^2}\right)du. $$

Therefore,

$$ {}^{\Psi}\hat{\Gamma}_p(x){}^{\Psi}\hat{\Gamma}_p(y) = 4\int_0^{\infty}\int_0^{\infty} u^{2x-1}v^{2y-1}{}_{\xi}\Psi_{\eta}\left(-u^2-\frac{p}{u^2}\right){}_{\xi}\Psi_{\eta}\left(-v^2-\frac{p}{v^2}\right)dudv. $$

In the above equality, taking u = r(cosθ) and v = r(sinθ) yields

$$ {}^{\Psi}\hat{\Gamma}_p(x){}^{\Psi}\hat{\Gamma}_p(y) = 4\int_0^{\frac{\pi}{2}}\int_0^{\infty} r^{2(x+y)-1}(\cos\theta)^{2x-1}(\sin\theta)^{2y-1} \times {}_{\xi}\Psi_{\eta}\left(-r^2(\cos\theta)^2-\frac{p}{r^2(\cos\theta)^2}\right) \times {}_{\xi}\Psi_{\eta}\left(-r^2(\sin\theta)^2-\frac{p}{r^2(\sin\theta)^2}\right)drd\theta, $$

Which completes the proof.

Theorem 2. The ${}_{\xi}\Psi_{\eta}$-beta function has the following integral representations:

$$ {}^{\Psi}\hat{B}_p(x,y) = 2\int_0^{\frac{\pi}{2}}(\sin\theta)^{2x-1}(\cos\theta)^{2y-1}{}_{\xi}\Psi_{\eta}\left(-p(\sec\theta)^2(\csc\theta)^2\right)d\theta, $$

$$ {}^{\Psi}\hat{B}_p(x,y) = \int_0^{\infty}\frac{u^{x-1}}{(1+u)^{x+y}}{}_{\xi}\Psi_{\eta}\left(-2p-p\left(u+\frac{1}{u}\right)\right)du, $$

$$ {}^{\Psi}\hat{B}_p(x,y) = (c-a)^{1-x-y}\int_a^c(u-a)^{x-1}(c-u)^{y-1}{}_{\xi}\Psi_{\eta}\left(\frac{-p(c-a)^2}{(u-a)(c-u)}\right)du. $$

Proof. Taking t = (sinθ)² in, we get

$$
\begin{aligned}
{}^{\Psi}\hat{B}_p(x,y) &= \int_0^1 t^{x-1}(1-t)^{y-1}\,{}_{\xi}\Psi_{\eta}\left(-\frac{p}{t(1-t)}\right)dt \\
&= 2\int_0^{\frac{\pi}{2}} (\sin\theta)^{2x-1}(\cos\theta)^{2y-1}\,{}_{\xi}\Psi_{\eta}\left(-p(\sec\theta)^2(\csc\theta)^2\right)d\theta.
\end{aligned}
$$

Taking t = $\frac{u}{1+u}$in , we get

$$
\begin{aligned}
{}^{\Psi}\hat{B}_p(x,y) &= \int_0^{\infty}\left(\frac{u}{1+u}\right)^{x-1}\left(\frac{1}{1+u}\right)^{y-1}\left(\frac{1}{1+u}\right)^{2}{}_{\xi}\Psi_{\eta}\left(-\frac{p}{\left(\frac{u}{1+u}\right)\left(\frac{1}{1+u}\right)}\right)du \\
&= \int_0^{\infty}\frac{u^{x-1}}{(1+u)^{x+y}}\,{}_{\xi}\Psi_{\eta}\left(-2p-p\left(u+\frac{1}{u}\right)\right)du.
\end{aligned}
$$

Taking t $=\frac{u-a}{c-a}$ in , we get

$$
\begin{aligned}
{}^{\Psi}\hat{B}_p(x,y) &= \int_a^c\left(\frac{u-a}{c-a}\right)^{x-1}\left(1-\frac{u-a}{c-a}\right)^{y-1}\frac{1}{c-a}\,{}_{\xi}\Psi_{\eta}\left(\frac{-p(c-a)^2}{(u-a)(c-u)}\right)du \\
&= (c-a)^{1-x-y}\int_a^c (u-a)^{x-1}(c-u)^{y-1}\,{}_{\xi}\Psi_{\eta}\left(-\frac{p(c-a)^2}{(u-a)(c-u)}\right)du,
\end{aligned}
$$

Which gives the result.

Research Problem

The central research problem in the study of generalizations of certain special functions, particularly hypergeometric functions, is driven by the pressing need to overcome the limitations of traditional mathematical tools. Special functions have historically played a pivotal role in solving a wide range of mathematical equations, but as contemporary scientific and engineering challenges become increasingly intricate and multidimensional, it has become evident that these functions have their constraints.The research problem can be succinctly summarized as the quest for more adaptable, versatile, and robust mathematical functions. These generalizations are needed to effectively handle problems that involve multivariate data, intricate mathematical structures, and complex relationships. Researchers are not only seeking to extend the range of functions that can be addressed but also to discover deeper connections and symmetries that underlie these generalizations.the research problem extends into the realm of practical applications. It seeks to create mathematical tools that can seamlessly bridge the gap between pure mathematics and applied sciences, enabling more efficient problem-solving in fields such as physics, engineering, statistics, and computer science.this research problem is a driving force for innovation and progress in mathematics. By pushing the boundaries of traditional special functions, researchers aim to unlock new possibilities and contribute to the advancement of mathematical theory, offering more elegant and efficient solutions to complex problems. In essence, it seeks to empower mathematicians and scientists with the tools needed to navigate the complexities of modern mathematics and its interdisciplinary applications effectively.

Extensions of Gauss and Confluent Hypergeometric Function

Extensions of the Gauss hypergeometric function and the confluent hypergeometric function refer to broader classes of mathematical functions that encompass and generalize the properties of these well-known special functions. These extensions are developed to address more diverse and complex mathematical problems in various fields, including mathematics, physics, engineering, and statistics. Here's a brief overview of these extensions:

1. **Generalized Hypergeometric Functions (${}_pF_q$):** The Gauss hypergeometric function, denoted as $_2F_1$, is a specific case of the generalized hypergeometric function ${}_pF_q$. In the generalized form, p and q can be any non-negative integers, and the parameters can be complex numbers. These functions are used to solve a wide range of linear differential equations and have applications in many areas of mathematics and physics.

2. **Confluent Hypergeometric Functions (Kummer Functions):** The confluent hypergeometric function, also known as the Kummer function or the confluent hypergeometric series, is a special case of the generalized hypergeometric function. It is typically denoted as ${}_1F_1(a; b; z)$. Extensions of the Kummer function involve variations in the parameters and can include the Tricomi confluent hypergeometric function, which generalizes the Kummer function to a wider class of differential equations.

3. **AppellHypergeometric Functions:**Appellhypergeometric functions, denoted as F_2, are extensions that encompass two variables and are often used in the context

of elliptic functions. These functions have applications in quantum mechanics, particularly in the study of hydrogen-like atoms.

4. **Generalized Hypergeometric Series with Matrix Arguments:** Extensions of hypergeometric functions can involve matrix arguments, allowing for the study of systems of differential equations with matrix coefficients. These functions are used in linear algebra and quantum mechanics, where matrix differential equations are common.

5. **q-Hypergeometric Functions:** q-hypergeometric functions are a generalization of the classical hypergeometric functions, introducing a parameter q. These functions are used in combinatorics, number theory, and quantum groups.

Extensions of Gauss and confluent hypergeometric functions provide mathematicians and scientists with more powerful and versatile tools to tackle complex mathematical problems in a wide range of applications. These functions often arise as solutions to differential equations and have diverse uses in areas such as quantum mechanics, special relativity, and statistical physics, among others.

Numerical values for MC.Fm(η,ζ;ξ;z) and MC.Φm(ζ;ξ;z) in Table 3.3 and Table 3.4 for m ranging from -2 to 2. Additionally, I will provide the values of the Gauss hypergeometric function and confluent hypergeometric function when m is equal to 0.

Conclusion

In conclusion, the exploration of generalizations of certain special functions, with a primary focus on extending the theory and applications of hypergeometric functions, emerges as a vital and dynamic field of mathematical research. This study is driven by the ever-increasing need to expand the mathematical toolkit to effectively address the multifaceted challenges presented by modern science, technology, and engineering.The significance of this research lies in its capacity to broaden the scope of mathematical problem-solving. Generalized special functions offer a versatile and adaptable framework that accommodates complex, multivariate data and provides innovative solutions to intricate mathematical problems. This adaptability has profound implications across a multitude of disciplines, including physics, engineering, statistics, and computer science.the exploration of generalizations unveils the inherent mathematical structures and symmetries within these functions, deepening our understanding of their underlying properties. This deeper comprehension not only enriches pure mathematics but also facilitates more efficient and elegant solutions to real-world problems.In the contemporary scientific landscape, the need for specialized mathematical tools is more pronounced than ever. The study of generalizations of special functions serves as a bridge between theoretical mathematics and practical applications, fostering innovation and progress in both realms. It reaffirms the enduring importance of special functions in mathematical analysis and positions them as indispensable assets in the pursuit of knowledge and advancement in various scientific domains.

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