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REVIEW OF THE EXPONENTIAL-INTEGRAL AND ASSOCIATED SPECIAL FUNCTION

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ABSTRACT

Used in generic q-exponentials to generalize full and unfinished gamma functions. We extend Special functions in probability, statistics and combinatorial terms are very important, and derive certain features. It may be derived from the generalized gamma feature approaching the usual gamma feature with certain q values such as q=0, 9 with a large value of q suffer from volatility.

Keywords: Exponential integral, q-Exponential Function, Special Function, Gamma Function, Ei Function

INTRODUCTION

The exponential integral egg is a unique feature in mathematics on the complex plane. The ratio of an exponential function to its input is defined as a certain integrative.

The exponential integral complement A(z) should be defined by

$$\text{Ein(z)} = \int_0^z \frac{1-e^{-t}}{t}\text{dt} = \sum_{n=1}^{\infty} \frac{(-)^{n-1} z^n}{n n!} \ (z \in \mathrm{C}) \quad (1)$$

And it's a whole function. Its link to the conventional exponential integral $\varepsilon_1(z)=\int_z^\infty t^{-1}e^{-1}\text{dt}$, valid in the cut plane | arg z| < π, [1]

$$\text{Ein(z)} = \log z + \gamma + \varepsilon_1(z), \quad (2)$$

Where γ = 0.5772156 . . . is the Euler-Mascheroni constant.

Mainardi and Masina have recently suggested to extend the Ein(z) function by substituting the exponential function (1) with the Mittag-Leffling one parameter function.

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{T\ (\alpha n+1)} (z \in \mathrm{C},\ \alpha > 0),$$

generalizing the exponential feature e^z. The function for every α> 0 was introduced in the cutting plane | arg z| < π

$$Ein_\alpha(z) = = \int_0^z \frac{1-E_\alpha(-t^\alpha)}{t^\alpha}\text{dt} = \sum_{n=0}^{\infty} \frac{(-)^n z^{\alpha n+1}}{(\alpha n+1)T\ (\alpha n+\alpha+1)}, \quad (3)$$

This simplifies to the Ein function when α = 1 (z). This function for $0 \leq \alpha \leq 1$ may be used physically in the investigation of the linear viscoelastic model creep features. Also explored a similar expansion of the sine and cosine integrals generalized. Plots of all these α ∈ [0, 1] functions have been provided

We look at a somewhat more general variant (3) based on the two-parameter function of Mittag-Leffler given by

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{T\,(\alpha n+\beta)} (z \in C, \alpha > 0),$$

Where β is going to be considered to be genuine. Then we will examine the extended complementary exponential integral

$$Ein_{\alpha,\beta}(z) = \int_0^z \frac{1-E_{\alpha,\beta}(-t^{\alpha})}{t^{\alpha}} dt = \sum_{n=1}^{\infty} \frac{(-)^{n-1}}{T\,(\alpha n+\beta)} \int_0^z t^{\alpha n-\alpha}\, dt$$

$$= z \sum_{n=0}^{\infty} \frac{(-)^n z^{\alpha n}}{(\alpha n+1) T\,(\alpha n+\alpha+\beta)} \quad (4)$$

When n−1 in the final summation is replaced by n. If β = 1 this decreases to (3)

$Ein_{\alpha},1(z) = Ein_{\alpha}(z)$.

This function's asymptotic growth is accomplished by α, β and fixed parameters for big complex z. This may be achieved via the use of the hypothesis for hypergeometric integrative functions as outlined in the reference. An intriguing aspect of the enlargement of $Ein_{\alpha,\beta}(x)$ for $x \to +\infty$ when α ∈ (0, 1] A logarithmic phrase appears every time

α $= 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\ldots\ldots\ldots$

LITERATURE REVIEW

KwaraNantomah (2021) HAL is an open access multidisciplinary archive for the deposit and distribution of scientific research materials, whether published or not. The papers may originate from French or foreign teaching and research institutions or from public or commercial research centers. The open multidisciplinary archive of HAL is intended for the repository and dissemination of scientific research materials, published or not, from French and foreign academic and research institutions, public and/or private laboratories.

Michael Milgram (2020) two literary identities are joined to produce an integrated Riemann equation, $\xi(s)$function and thus$\zeta\,(s)$ indirectly. The equation has a variety of basic characteristics, the most noteworthy of which is the useful derivative flow $\zeta\,(s)$ anyplace on a line somewhere else on the complicated plane, on a crucial strip with its values. Both of these are an analytical expression$\zeta\ (\sigma + it)$, in the asymptotic everywhere (t→ ∞)Critical strips and an approximate solution may be obtained in the hypothesis of Riemann is true of this. The solution promises a simple yet strong connection between the real and the imaginaryμ and t equivalents.

Francesco Mainardi and Enrico Masina (2018) in this article we examine the characteristics of the Schelkunoff modification and extend it using the Mittag-Leffler function. Through its full monotonicity, we get a new unique function which may become important for linear viscos-elastics. We also examine the generic functions of the sinus and cosine.

Francesco Mainardi(2018) we are presenting a novel rheological model according to a real parameter ν∈[0,1], this lowers the Maxwell v=0 body and the Becker v=1 body. The related creep rule is stated in a comprehensive manner, replacing and generalizing the Becker model's exponential function with Mittag –Leffler order function¨. The associated non-dimensional cracking function and rate for various values¨are then examined to demonstrate a move from the conventional Maxwell to the Becker body as time functions. In addition, we may approach the relaxant function by numerically solving an integrated Volterra second type of equation based on the classical theory of linear viscoelasticity.

IvanoColombaro, (2017) In this article, we investigate a wide range of linear viscoelastic models expressed in the Laplace domain using the proper ratios of modified contiguous Bessel functions. The Dirichlet series shows these functions in time. The relaxation module and conformity with result in an endless, discreet spectrum of delays and relaxation time correspondingly. In reality, we obtain a viscoelastic class according to a real parameter ν>−1Such models have rheological characteristics for short periods, similar to the fractional Maxwell model (ordering 1/2) and the conventional Maxwell model for long periods.

THE EXPONENTIAL INTEGRAL AND ITS FUNCTION

We may start directly with the mathematical formulation of the so-called Exponential Integral Function.

$$\mathrm{Ei}(x) = \int_{-\infty}^{x} \frac{e^t}{t}\, dt \quad x \in \mathrm{W}^1$$

We can see, then, that the exponential integral is defined as a specific.The connection between the exponential function and the argument integral.

The Risch algorithm may be used to demonstrate that this integral is not a basic function, that is to say, there is not a primitive Ei(x) in elementary functions. Such a function has a pole at t=0, therefore we interpret this integral as the main value of Cauchy:

$$\mathrm{Ei}(x) = \lim_{\alpha \to 0} \left[\int_{-\infty}^{\alpha} \frac{e^t}{t}\, dt + \int_{\alpha}^{x} \frac{e^t}{t}\, dt \right]$$

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We can better define $\mathrm{Ei}(x)$ in a parity transformation

$$\begin{cases} \mathrm{t} \rightarrow -\mathrm{t} \\ \mathrm{x} \rightarrow -\mathrm{x} \end{cases}$$

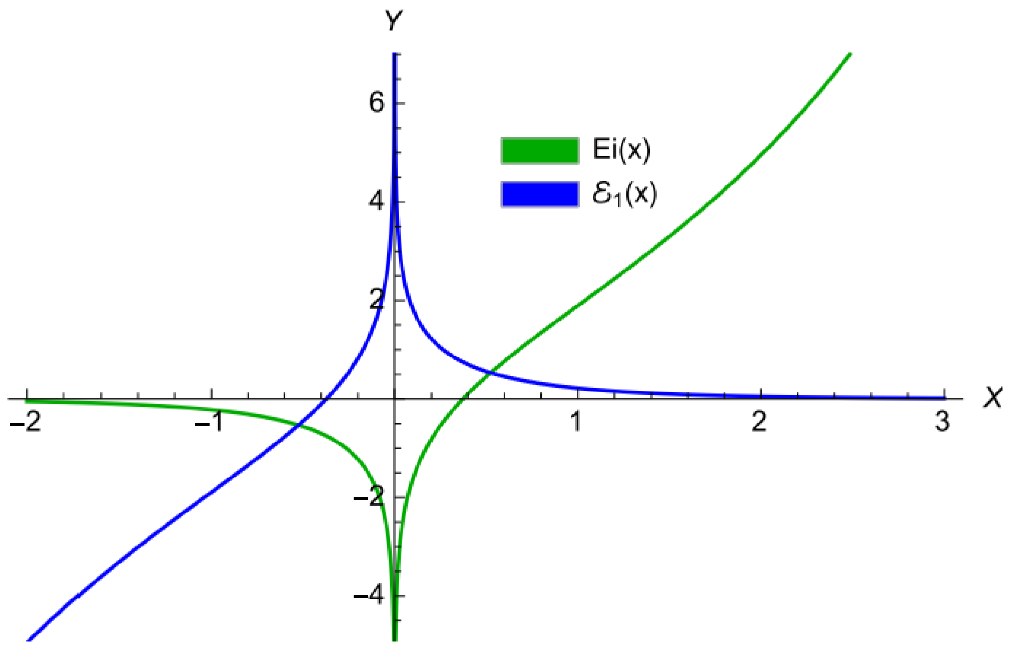
one gets

$$\mathcal{E}_1(x) = -\mathrm{Ei}(-x) = \int_x^{+\infty} \frac{e^{-t}}{t}\, dt$$

If the argument accepts complex values, it becomes confusing to define the integral, because of the branch points at 0 and ∞. Introducing the complex variable z = x + y, we can set the exponential integral in the complex plane using the following notation:

$$\mathcal{E}_1(z) = \int_z^{+\infty} \frac{e^{-t}}{t}\, dt \qquad |\arg(z) < \pi$$

The following figure may be useful to understand the graphical behavior of the two functions.



We can immediately write down some useful known values:

$\mathrm{Ei}(0) = -\infty$	$\mathrm{Ei}(-\infty) = 0$	$\mathrm{Ei}(+\infty) = +\infty$
$\mathcal{E}_1(0) = +\infty$	$\mathcal{E}_1(+\infty) = 0$	$\mathcal{E}_1(-\infty) = -\infty$

It's actually simple to find the values of $\mathcal{E}_1(x)$ from $\mathrm{Ei}(x)$ (and vice versa) by using the previously written relation:

$$\mathcal{E}_1(x) = -\mathrm{Ei}(-x)$$

We notice that the function $\mathcal{E}_1(x)$ is a monotonically decreasing function in the range (0, ∞). The function $\mathcal{E}_1(z)$ is actually nothing but the so-called Incomplete Gamma Function:

$$\mathcal{E}_1(z) \equiv \mathrm{T}(0, \mathrm{z})$$

Where

$$\Gamma\,(\mathrm{s},\mathrm{z}) = = \int_z^{+\infty} t^{s-1} e^{-t}\ dt$$

Indeed, by putting s = 0 we immediately find $\mathcal{E}_1(z)$.

By introducing the small Incomplete Gamma Function

$$\gamma(s\,, z) = \int_0^z t^{s-1} e^{-t}\ dt$$

We can put down a very clear, sometimes helpful and direct relationship between the three Gamma functions:

$$\mathrm{T}\,(\mathrm{s}, 0) = \gamma\,(\mathrm{s}, \mathrm{z}) + \mathrm{T}\,(\mathrm{s}, \mathrm{z})$$

Now let's return to the Exponential Integral.

Let's make a naïve variable change

$t \rightarrow z\,u$ $\qquad$ $dt = z\,du$

Step by step we get:

$$\int_z^{+\infty} \frac{e^{-t}}{t}\ dt \ \rightarrow \int_1^{+\infty} \frac{e^{-zu}}{zu} z\ du = \int_1^{+\infty} \frac{e^{-zu}}{u} du$$

In this way we define the General Exponential Integral:

$$\mathcal{E}_n(z) = \int_1^{+\infty} \frac{e^{-zu}}{u^n} du n \in \mathrm{R}$$

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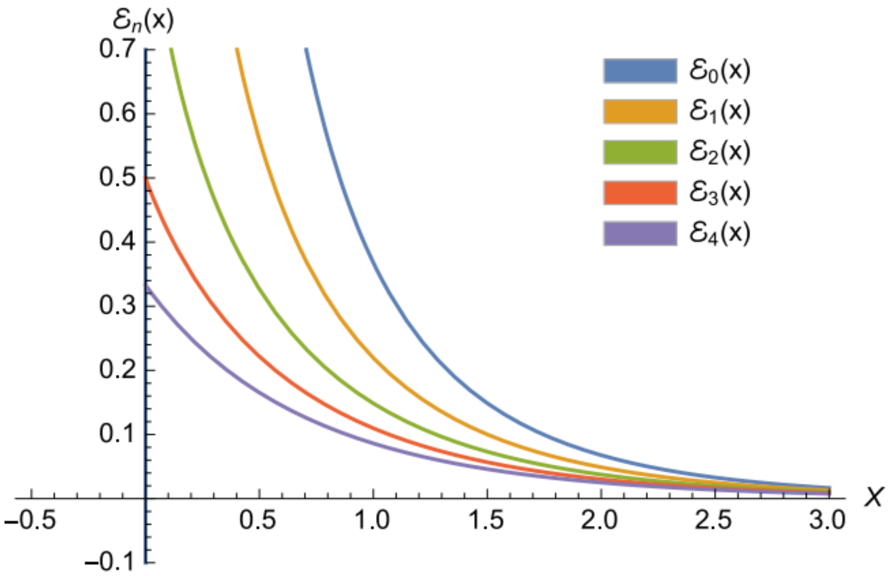
with the particular value

$$\mathcal{E}_n(0) = \frac{1}{n-1}$$

We can show the behavior of the first five functions $\mathcal{E}_n(x)$, namely for n = 0,... , 5.



GENERALIZED GAMMA FUNCTION

Definition of the generalized gamma function by integral, by means of the q-exponential distribution

$$\Gamma q(z+1) = \int_0^\infty x^z {e_q}^x \, dx, \quad \textbf{(5)}$$

Where q∈(0,1] and z∈ℂ and$\Re$e{z}>0. In the limit q→1, we have

Γq(z)=Γ1(z)=Γ(z) and Γ(n)=(n−1)!

For n∈ℕ and Γ(z+1)=zΓ(z) for z∈C.

Since${e_q}^{x-1} \ll x^{z-1}$,when we can write x is positive and x is(0,1)

$$\left|\int_0^1 {e_q}^x x^{z-1} \, dx\right| < \left|\int_\in^1 x^{z-1} \, dx\right| = \frac{1}{z} - \frac{\epsilon^z}{z} \quad (6)$$

and the integral for x>0 for 1/x is restricted.

By fixing and reducing x the integral value grows monotonously, i.e.

$$\int_0^1 e_q{}^x x^{z-1}\, dx = \lim_{\epsilon\to 0} \int_\in^1 e_q{}^x x^{z-1}\, dx, \exists\forall_x > 0 \qquad (7)$$

$e_q{}^{ix}$ Presents the properties$[e_q{}^{ix}] = e_q{}^{-ix}$,q-exponential functions are deformed by means of a real parameter q in the conventional exponential function

$$e_q{}^x = \begin{cases} [1+(q-1)x]^{\frac{1}{q-1}}, & -\infty < x \le 0, \\ [1+(1-q)x]^{\frac{1}{1-q}}, & 0 \le x < \infty, \end{cases} \qquad (8)$$

The opposite of the q-exponential functions is the lnq(x) function, defined as the q-logarithm

$$\text{Inq}(x) = \begin{cases} \frac{x^{q-1}-1}{q-1}, & 0 < x \le 1, \\ \frac{x^{1-q}-1}{1-q}, & 1 \le x, < \infty. \end{cases} \qquad (9)$$

The equation formulas (8) apply exclusively to q∈(0,1], x and q are separate mathematically at this interval. There are two equal methods to provide the entire definitions: either one correctly transforms the whole term of Uses the interval q∈(0,1].A single expression and deformation parameter changes interval may only be considered, as is the case with the preceding equation

$$e_q{}^x = [1+(1-q)x]^{\frac{1}{q-1}}, \begin{cases} -\infty < x \le 0, & q \in [1,2), \\ 0 \le x < \infty, & q \in (0,1]. \end{cases}$$

$$\text{Lnq}(x) = \frac{x^{1-q}-1}{1-q} \begin{cases} 0 < x \le 1, & q \in [1,2), \\ 1 \le x < \infty, & q \in (0,1]. \end{cases} \qquad (10)$$

The parameter q is the non-additive degree. Therefore we get a generalized gamma function in the integral equation (5) and utilizing the concept of q-exponential

$\Gamma q(p+1)$ (11)

$$= \frac{p(p-1)(p-2)(p-3)\times\cdots\times[p-(p-1)]}{(2-q)(3-2q)(4-3q)(5-4q)\times\cdots\times[p+2-(p+1)q]} \int_0^\infty (e_q{}^{-x})^{(p+2)(1-q)+q}\, dx,$$

Where Γ(p+1)=p!For p∈N and Γ (z+1)=zΓ(z), z∈C. Thus, we followed the recurrence relation for the generalized gamma function provided by the standard factor function

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$$T_{q}(z+1)=\frac{z\Gamma(z)}{\prod_{j=1}^{p}[j+2-(j+1)q]}, \quad (12)$$

As a result, we may get the q-factorial expression, [p]q!

$$[p_q]! = \frac{p!}{\prod_{j=1}^{p}[j+2-(j+1)q]} \quad (13)$$

Where p∈N.

We also get the incomplete gamma functions

$$T_q\ (a,x)= \int_x^{\infty} z^{a-1} {e_q}^{z}\, dz. \quad (14)$$

$$\gamma_q\ (a,x) = \int_0^x z^{a-1} {e_q}^{z}\, dz, \quad (15)$$

With ℜe(a)>0, where

$$T_q\ (a,x)+\gamma_q\ (a,x)=T_q\ (a). \quad (16)$$

We have the following generalized features, if the unfulfilled gamma function is involved.

$$\mathrm{erfc}_q(x) = \frac{1}{\sqrt{\pi}}\, \gamma q\ (1/2, x^2) \quad (17)$$

This is the generalized additional error function

$$E_{qn}\ (x)= \int_1^{\infty} \frac{{e_q}^{-xt}}{t^n}\, dt, \quad (18)$$

If the exponential integral function is defined generalizedEq1(x)=−Eqi(−x) as

$$E_{q1}(x) = \int_{-\infty}^{x} \frac{{e_q}^{t}}{t}\, dt. \quad \mathbf{(19)}$$

Figure 1 shows the gamma function in general —functional q for q = 0,9 and the gamma function in the standard —functional, which corresponds in a situation of q = 1. The q(z) graph varies significantly with the q value as seen.

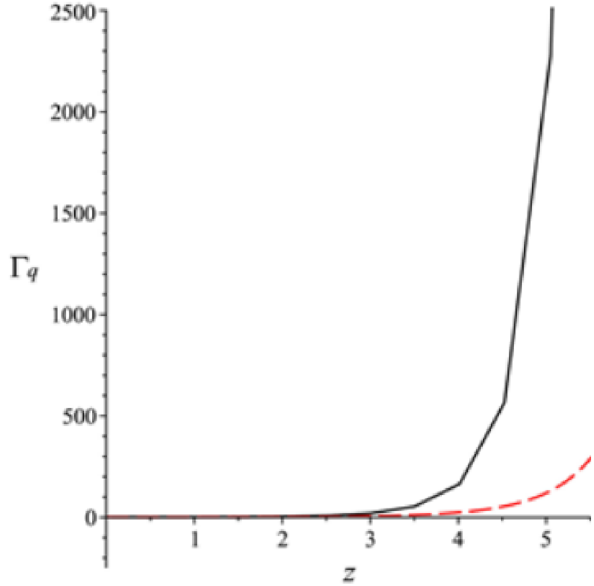

Figure 1. Plot of the generalized gamma function Γq(z)

In illustration 2. This is because the q-exponential function represents the function family (one for each q inside the interval (0,1) while the q=1(ex) situation only corresponds to one exponential function of the type of q. The q-gamma function indicates an approach nearer than q-exponentials to ordinary exponential for various q values.

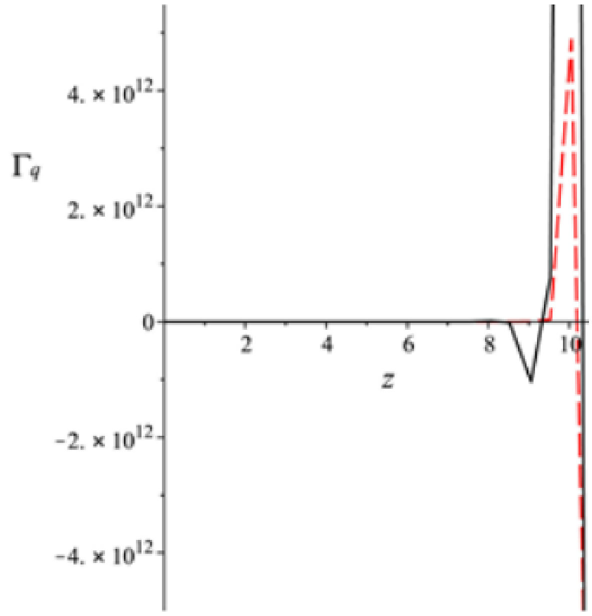

Figure 2. Plot of the generalized gamma function Γq(z) for multiple q values

The generalized incomplete q- gamma function is provided as a result of the q-exponential definition.

$$T_{q\ (a\ ,x)} = \frac{x^{a-1}}{2-q}\left(e_q{}^{-x}\right)^{2-q} + \frac{a-t}{2-q}\, Tq\ (a-1, x) \quad \textbf{(20)}$$

And

$$\gamma_{q}\ (a\ ,x) = \frac{x^{a-1}}{q-2}\left(e_q{}^{-x}\right)^{2-q} + \frac{a-t}{2-q}\gamma q\ (a-1,x). \quad \textbf{(21)}$$

Furthermore, the exponential integral function is also available generally

$$E_{q1}(x) = \frac{1}{2-q}\left(e_q{}^{x}\right)^{\frac{2-q}{1-q}}. \quad \textbf{(22)}$$

Finally, we have a widespread integrated logarithm

$li_q\ (x) = E_q i\ (In_q\ (t))$, given by

$$li_{q1}\ (x) = \int_0^x \frac{dt}{In_q\ (t)}. \quad \textbf{(23)}$$

Consequently, we get $li_q\ (x)$ given as

$$li_{q1}\ (x) = \int_0^x \frac{dt}{In_q\ (t)} = (1-q)\int_0^x \frac{dt}{t^1 - q - 1}$$

$$= -\int_0^x dt \sum_{n=0}^{\infty} (t^1 - q)^n$$

$$= -\sum_{n=0}^{\infty} \frac{x^{n(1-q)}}{n(1-q)} \quad (24)$$

where $|x|<1$.

CONCLUSION

An apparently novel extension of the exponential integral E1 in the Gamma function is given and a more general expansion is demonstrated. This latter expansion is proven here by viewing it as a "theorem for multiplication." The complementary gamma function has a supplementary result not shown and may be used to produce an extension linking E1 to many parameters. A general technique for turning a power series into gamma expansion is explained.

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