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Sensitivity Analysis of the Mathematical Model in a Complex System

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Abstract: The process of building a mathematical model of a complex system frequently involves assessing the impact of inputs (arguments, factors) on the output (response), identifying significant correlations between the variables, and condensing the model by reducing the number of inputs. These assignments deal with Sensitivity Analysis issues in mathematical models. In order to evaluate the contribution of changes to a function's variables to changes in the output, the author suggests an alternate method based on the application of Analysis of Finite Fluctuations and the Lagrange mean value theorem. The approach is examined in the article using a class of fully connected neural network models as an example. Sensitivity analysis yields a set of sensitivity measurements for every input. The consistency of the suggested approach is demonstrated by comparing the disclosed method. An examination is conducted on the computational resilience of the process used to determine input sensitivity measures.

Keywords: sensitivity analysis; neural networks; sensitivity measures

1. Introduction:

Sensitivity Analysis (SA), the study of how input uncertainty affects output uncertainty, is a crucial component of system research. A task such as that could be useful in solving the following issues, depending on the goal: (1) assessing the robustness of a system's or model's results; (2) comprehending the relationships between inputs and outputs; (3) minimizing uncertainty by selecting the most important inputs; (4) identifying errors in a system or model by defining unexpected relationships between inputs and outputs; (5) streamlining a model by eliminating inputs that do not significantly impact outputs; (6) making a model more interpretative by determining an understandable explanation of inputs; and so on. Numerous approaches for sensitivity-measuring mathematical model sensitivity can be put forth, depending on the instruments employed. Only when the model has a predetermined structure can the others be used; some are universal. The SA taxonomy proposed by Saltelli [2] splits all methods into two categories: local and global sensitivity analysis. Let us need the vector variable $X = (x_1, x_2, \dots, x_n)$ moving the scalar function y by the black-box relationship

$$y = f(x_1, x_2, \dots, x_n) = f(X) \dots \dots . . (1)$$

Finding out how uncertain an output is in relation to a change in one of its inputs forms the basis of the first set of methods (local SA). However, this understanding of sensitivity has two obvious drawbacks: first, the sensitivity assessment result is dependent on the range of selected x_i values in the case of a non-linear model (1); second, in the case of existing factor interactions, the change to the partial derivative $\partial y/\partial x_i$ is dependent on other factors corresponding to x_i in addition to the selected x_i values. This indicates that methods for the local SA produce sufficient and precise outcomes in situations where Model (1)'s structures are somewhat constrained, which restricts the use of these methods.

$$S_i = \frac{V_{x_i}\left(E_{x\sim i}(y \mid x_i)\right)}{V(y)}, \dots \dots \dots (2)$$

where$V(y)$ is the unconditional variance of y, which is obtained when all factors x_i are allowed to vary; $E_{x\sim i}(y \mid x_i)$ is the mean of y when one factor is fixed. When Model (1) is nonlinear and its components are independent of one another, Global SA should be used. It should be mentioned that sensitivity measuring in applied problems has drawn more attention throughout the past ten years. Various methods and strategies are employed, depending on the specific issue at hand. One of the most prominent modern domains is medicine; in the example study, the key electrical and structural parameters of the arm arteries are determined by applying a parametric local SA approach to a linear elastic lumped parameter model. In order to determine the primary factors that most significantly influence the absorption and distribution of medications and nanoparticles in the various organs inside the human body, the study [4] uses a partial-rank correlation coefficient to do global statistical analysis. Environmental studies are a field in which

SA is also extensively used. A comparison of methods that were available at the time to determine which environmental model factors had the most influence is provided in the case study. The goal of the study is to discover thresholds to identify insensitive input components for environmental models and to solve the sample size choice problem. In the context of environmental modelling, the study offers guidance on selecting suitable emulation modelling techniques for SA in order to lessen the complexity of the model in question. The algorithm suggested in, which is based on Analysis of Finite Fluctuation, is examined in this work. This approach is a global SA strategy that involves figuring out the precise values of factors' finite changes' effects on the finite change of the model output, as well as obtaining a parameter of the intermediate point α in the Lagrange mean value theorem. Our method reduces the accuracy of the results obtained compared to previous ways because it does not employ an approximation of the original model. It also allows factors to be interconnected; the first partial derivatives are the only restriction. Additionally, our method considers all relevant data regarding the original model's parameters and factors, ensuring the analysis's results are reliable.

2. State-of-the-Art in Sensitivity Measures

Additionally, we offer a few methods and strategies for both local and global sensitivity analysis, which are mostly employed to address specific issues.

2.1 Local Techniques

Local SA contains of estimating

$$A_i = \frac{\partial y}{\partial x_i}\left(x_1^{(0)}, x_2^{(0)}, \dots, x_n^{(0)}\right) \dots\dots\dots\dots\dots\dots..(3)$$

whichtypifies how perturbation of x_i affects the productivity$y$ near the value $X^{(0)}$.A mutual approach to evaluate estimation of (3) based on the first partial derivatives usually refers to the "one at a time" (OAT) line. Let x^*stand the nominal value of the input i; define $y_i^{\max} = f(x_1^*, x_2^*, \dots, x_i^{\max}, \dots, x_n^*)$ as the model production where altogether factors are at triflingstandardsbut the factor i, which is set to its supreme. The OAT sensitivity amount can be defined as

$$\Delta_i = \frac{y_i^{\max} - y_i^{\min}}{x_i^{\max} - x_i^{\min}}$$

where$y_i^{\min}$ is the model productivity where all factors stand at nominal values excluding the factor i, which is customary to its minimum. The OAT tactic keeps all factors fixed excluding the one that is below perturbation.Another prevalent local SA tactic is the Morris method. Let $r$stand the integer of OAT realizations; we generate the discrete picture of the input space in a d-dimensional grid through$n$ levels thru input; $E_j^{(i)}$ is the elementary effect of the j-th variable gained at the i-th repetition, distinct as:

$$E_j^{(i)} = \frac{f\left(x^{(i)} + \Delta e_j\right) - f\left(x^{(i)}\right)}{\Delta},$$

2.2 Global Techniques

Methods based on the study of linear models make up the first category of approaches. Assume that the studied model's inputs and outputs are readily available and that a linear fit to the current relationship (1) is achievable. A fitted linear model is intended to be used in certain methods for evaluating the sensitivity measurements. The often used indices in this instance are:

- Pearson correlation coefficient $\rho\left(x_j, y\right)$;
- Standard regression coefficients (SRC)

$$SRC_j = \beta_j \sqrt{\frac{\mathrm{Var}\left(x_j\right)}{\mathrm{Var}(y)}}$$

whereβ_j are regression coefficients obtained during its parametric identification;

- Partial correlation coefficient (PCC)

$$PCC_j = \rho\left(x_j - \hat{x}_j, y - \hat{y}\right).$$

When there is no way to determine Model (1) using a linear model or when this structure is non-monotonic, sensitivity can be evaluated by breaking down the output variance. One way to describe (1) as a sum of elementary functions is if it is a square-integrable function defined on the unit hypercube $[0,1]^n$.

$$f(X) = f_0 + \sum_{i=1}^{n} f_i(x_i) + \sum_{i<j}^{n} f_{ij}\left(x_i \cdot x_j\right) + \cdots + f_{12\ldots n}(X) \ldots\ldots\ldots\ldots\ldots..(4)$$

Conferring to [28], under the complaint

$$\int_0^1 f_{i_1\ldots i_s}\left(x_{i_1},\ldots,x_{i_s}\right)dx_{i_k} = 0, \quad 1 \le k \le s, \quad \{i_1,\ldots,i_s\} \subseteq \{1,\ldots,n\}$$

the expansion (4) is unique.Let us have the random vector $X \in R^n$ with mutually independent components and the output y connected to X by Model (1). In this case, a functional decomposition of the variance is available:

$$\mathrm{Var}(y) = \sum_{i=1}^{n} D_i(y) + \sum_{i<j}^{n} D_{ij}(y) + \cdots + D_{12\ldots n}(y) \ldots\ldots\ldots\ldots\ldots.(5)$$

$$D_i(y\mathrm{Var}[E\left(y \mid x_i, x_j\right)] - D_i(y) - D_j(y)$$

$$S_i = \frac{D_i(y)}{\mathrm{Var}(y)}, \quad S_{ij} = \frac{D_{ij}(y)}{\mathrm{Var}(y)}, \ldots$$

According to [30],

$$S_{T_i} = S_i + \sum_{i<j} S_{ij} + \sum_{i\neq j,k\neq i,j<k} S_{ijk} + \cdots = \sum_{l\in\#i} S_l \ldots\ldots\ldots\ldots\ldots(6)$$

where$\#i$ are completely the subsections of $\{1,\ldots n\}$with$i$. In repetition, when n is large, only the key effects (2) and the entire effects (3) are figured, thus giving reliable information about model sensitivities.

3. Analysis of Finite Fluctuations as an Method to Sensitivity Measuring

This section is enthusiastic to the introduction of the novel technique and to the enquiries of its stability.

3.1 Technique Description

In Mathematical Analysis, a model of these linkages exists if fluctuations are modest. When the model's y=f(x) function is defined as continuous and differentiable in a closed domain, the response's approximate relationship to the arguments' tiny fluctuations is

$$\Delta y = f(X^0 + \Delta X) - f(X^0)$$
$$= f\left(x_1^{(0)} + \Delta x_1, \dots, x_n^{(0)} + \Delta x_n\right)$$
$$-f\left(x_1^{(0)}, \dots, x_n^{(0)}\right) \approx \sum_{i=1}^{n} \frac{\partial f\left(X^{(0)}\right)}{\partial x_i} \cdot \Delta x_i.$$

However, fluctuations may not be thought of as minuscule values for some practical difficulties, but rather as finite ones. For functions that are defined, continuous, and have continuous partial derivatives inside a closed domain, this is the Lagrange mean value theorem. The way it is formulated is as follows:

$$\Delta y = \sum_{i=1}^{n} \frac{\partial f\left(X^{(m)}\right)}{\partial x_i} \cdot \Delta x_i, \dots \dots \dots \dots . . (7)$$
$$X^{(m)} = \left(x_1^{(m)}, \dots, x_n^{(m)}\right),$$
$$x_i^{(m)} = x_i^{(0)} + \alpha \cdot \Delta x_i, \quad 0 < \alpha < 1.$$

value of α.Thus, the increment of the output can be defined, on the one hand, as the difference in the new and previous values of the outputs and, on the other hand, by the Lagrange theorem, i.e., the following equation can be composed and solved with respect to the parameter α:

$$y^{(1)} - y^{(0)} = \sum_{i=1}^{n} \frac{\partial y}{\partial x_i}\left(\dots, x_i^{(0)} + \alpha \cdot \Delta x_i, \dots\right) \cdot \Delta x_i \dots \dots \dots \dots . (8)$$

which allows estimate of the so-called factor loadings towards obtain a perfect of the form

$$\Delta y = \sum_{i=1}^{n} \frac{\partial y}{\partial x_i}\left(\dots, x_i^{(0)} + \alpha \cdot \Delta x_i, \dots\right) \cdot \Delta x_i = A_{x_1}\Delta x_1 + A_{x_2}\Delta x_2 + \dots + A_{x_n}\Delta x_n \dots . . (9)$$

The way above is frequent$m$ times (where m is the number of accessible observations); the numerical consequences of the examination have to be an average of to construct the sensitivity measure.

3.2. Sensitivity Measure

The following steps are part of the iterative algorithm used to create this estimation:

- Determine the sample mean (the procedure typically starts with the median).
- Calculate the separations between each sample element and the computed mean. The sample elements are given varying weights based on these distances, which are then taken into consideration while recalculating the mean. Because of the way the weight function works, observations that deviate significantly from the mean don't add much to the weighted mean.

Let $A_{x_i} = \{A_{x_i}^1, A_{x_i}^2, \dots, A_{x_i}^m\}$ be a sample from the intended factor loadings aimed at the input $x_i, M_{A_{x_i}}$ is the example median of the set A_{x_i}, and S is the taster median of the set $\{|A_{x_i}^1 -$ $MAxi,\ldots,Axin-MAxi$. Aimed at each element $Axik(k=1,2,\ldots,n)$ of the sample, the deviation as of the mean is designed

$$u_k = \frac{A_{x_i}^k - M_{A_{x_i}}}{c \cdot S + \varepsilon}$$

where ε is a modest value that serves mostly to exclude the possibility of division by zero, and c is a parameter that controls how sensitive the estimate is to outliers. We employ a bi-quadratic function of the form to determine the weight of each observation in the sample.

$$w(u) = \begin{cases} (1-u^2)^2, & |u| \le 1 \\ 0, & |u| > 1 \end{cases}$$

$$T_{A_{x_i}} = \frac{\sum_{k=1}^{m} w(u_k) \cdot A_{x_i}}{\sum_{k=1}^{m} w(u_k)} \dots\dots\dots\dots (10)$$

The symmetric confidence intermission is prearranged by

$$T_{A_{x_i}} \pm t_{df}^{(1-\alpha/2)} \cdot \frac{S_{A_{x_i}}}{\sqrt{m}} \#(11)$$

$$S_{A_{x_i}} = \sqrt{m} \cdot \frac{\sqrt{\sum_{|u_k|\le 1} \left(A_{x_i}^k - T_{x_i}\right)^2 (1-u_k^2)^4}}{\left|\sum_{|u_k|\le 1} (1-u_k^2)(1-5u_k^2)\right|} \dots\dots\dots.(11)$$

where$t_{df}^{(1-\alpha/2)}$ is the $(1-\alpha/2)$-quantile of the Student's t-distribution with the number of degrees of freedom $df = \max(0{,}7 \cdot (m-1),1)$.

4. **Numerical Example**

In this segment, we offer instances of utilizing the recently suggested method to evaluate sensitivity metrics of a neural network model's components. The first example is synthetic and is built on the popular extended data set from the R language's NeuralNetTools package, while the second example makes use of actual healthcare data indicators from the Lipetsk region's Compulsory Medical Insurance Fund. A number of computer experiments were carried out to evaluate the suitability of the outcomes attained with the suggested method. The data used was the neuraldata data set from the R data processing environment's NeuralNetTools package. We can obtain 1999 finite response increments and their accompanying justifications because the set has 2000 realizations. Three independent and two dependent variables make up the given data set; at this point, only one dependent variable has been used as an output for model construction and study. In numerical experiments, the approach presented in Table 1 was followed to raise the number of independent variables to 15.A sequence of random numbers with various statistical distributions (Weibull and normal) and values of these distribution parameters are also represented by the independent variables that were acquired. Figure 1 box-plots illustrate how the spreads for the independent and dependent variables differ.

Table 1: Test data characteristics.

Input Factor	Characteristic	Input factor	characteristic
x_1	$N(0,1)$	x_8	$x_2 - \exp(x_6)$

x_2	$N(0,1)$	x_9	$x_3 + \sin(x_7)$
x_3	$N(0,1)$	x_{10}	$x_1 - \cos(x_8)$
x_4	$x_1 - \exp(x_2)$	x_{11}	$x_2 + \tan(x_9)$
x_5	$x_2 + \sin(x_3)$	x_{12}	$x_3 - \exp(x_{10})$
x_6	$x_3 - \cos(x_4)$	x_{13}	$x_1 + \sin(x_{11})$
x_7	$x_1 + \tan(x_5)$	x_{14}	$x_2 - \cos(x_{12})$
x_8	$x_2 - \exp(x_6)$	x_{15}	$x_3 - \tan(x_{13})$

For the numerical experiment, a neural network model with one hidden layer made up of three neurons and the following topology was used:

$$y = \phi_1\left(b_0 + \sum_{i=1}^{3}\left(b_i + w_i \phi_2\left(\sum_{j=1}^{15} w_{ij}\, x_j\right)\right)\right), \dots\dots\dots\dots\dots\dots (14)$$

everywhere$y \in \mathbf{R}$ is the comeback value; $X = (x_1, x_2, \dots, x_{15})$ is a vector of factors; w_i and w_{ij} stand weight coefficients of the yield and hidden layers, individually; b_0 and b_i are partialities of the production and hidden layers, individually; $\phi_1(\text{ net }) = \phi_2(\text{ net }) = 1/(1 + \exp(-\text{ net }))$ are logistic start functions.

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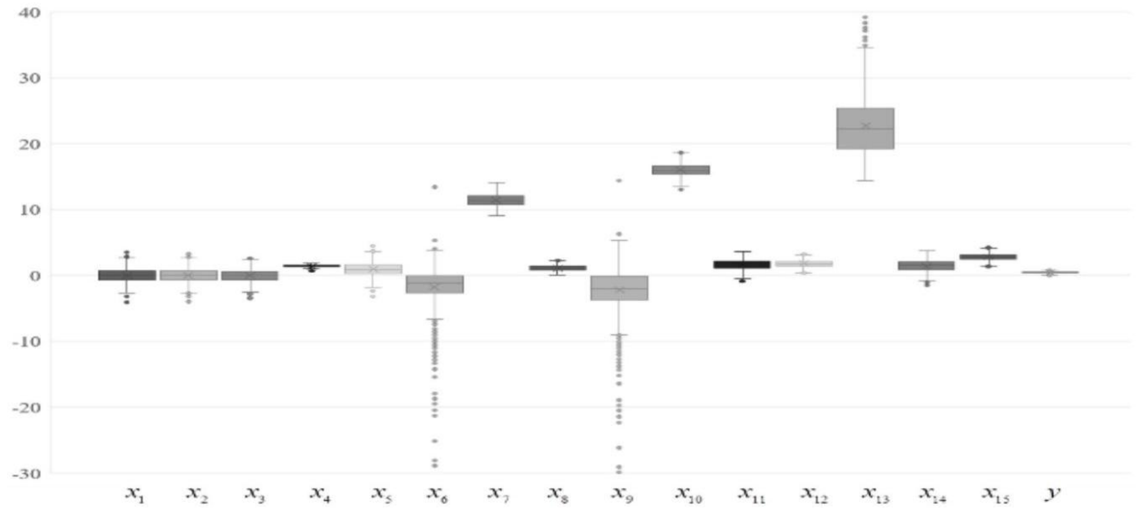

Figure 1: Variations to output and input factors

We can proceed with numerical experiments to determine the sensitivity of the model by using the model with an MAE of 0.092% for the first data set and model, indicating great accuracy of the proposed neural network model (4).

5. Discussion of Results

The calculated values of the sensitivity measures (Section 3) (point estimates obtained as weighted averages (2) and interval estimates (4)) for the described data set are presented in Figure 2. According to the obtained results, the variables $x_1, x_2, x_3, x_5, x_{12}$, and x_{15} have the highest sensitivity. It should also be noted that the approach for analyzing the sensitivity of the model based on using Analysis of Finite Fluctuations, in addition to the strength of the influence of factors, assesses the direction of this influence. Thus, for Model, the factors x_3, x_5, and x_{12} have a positive influence on the output y, while the factors x_1, x_2, and x_{15} influence negatively.Figure 2 displays the computed values of the sensitivity measures for the given data set (point estimates produced as weighted averages and interval estimates. The variables with the highest sensitivity are x_1, x_2, x_3, x_5, x_{12}, and x_{15}, per the results that were obtained. It should be mentioned that the method for evaluating the model's sensitivity based on the Analysis of Finite Fluctuations approach evaluates the direction of the influence of the elements in addition to their strength. The factors x_3, x_5, and x_{12} have a positive influence on the output y in Model, whereas the components x_1, x_2, and x_{15} have a negative influence.Figure 3 compares the evaluated sensitivity measures by utilizing the School sensitivity index algorithm, determining the weighted average for each factor loading, and assessing final variations in the factor loadings in line with the suggested approach. After examining the collected data, we may conclude that the outcomes of the two approaches are comparable. It should be noted that all technique results were given as percentages because different methods employed different scales.The above-described method was used to investigate the computational stability of the suggested strategy (Mann-Whitney-Wilcoxon statistics were calculated). The Mann-Whitney-Wilcoxon criterion calculated values are greater than the tabular values for all analyzed input components. Therefore, in the case of random fluctuations, the hypothesis regarding the existence of a shift between the mean values in the samples derived from the estimations of the influence of factors on the model output can be rejected.

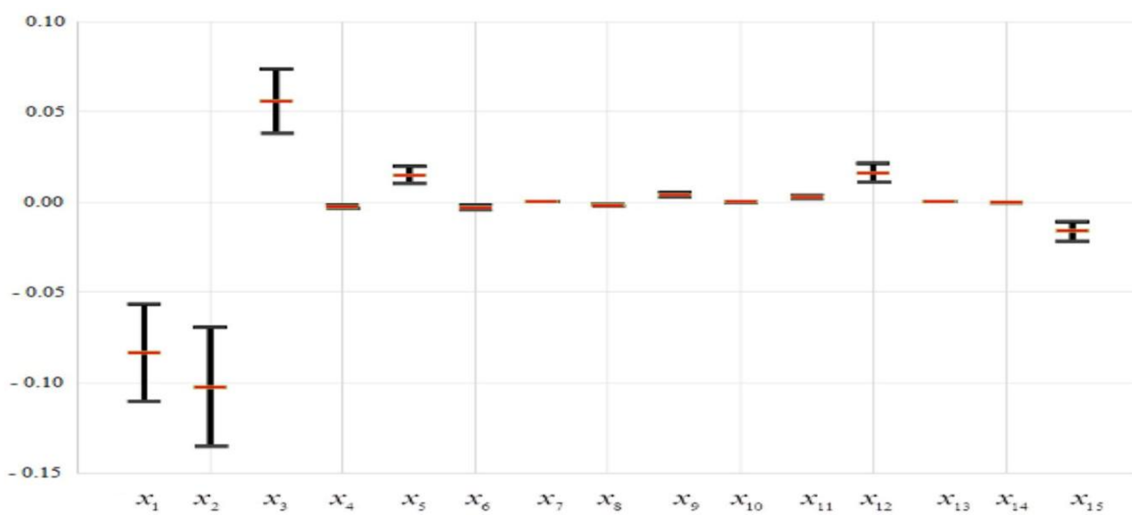

Figure 2: Assessed sensitivity measures of Model

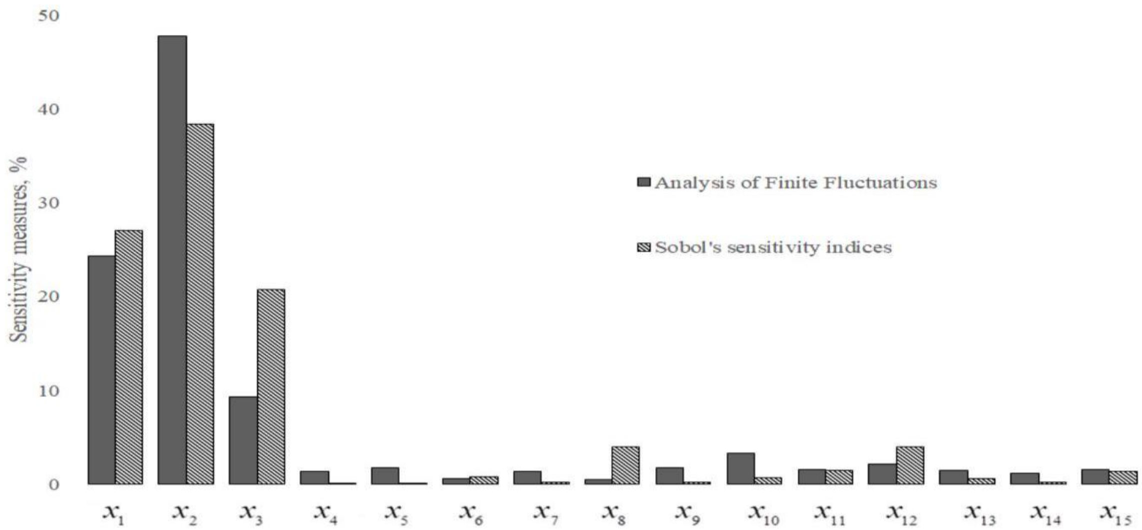

Figure 3: Comparison of assessed sensitivity measures.

6. Conclusions

The article provides taxonomy of current sensitivity evaluation techniques based on mathematical model factors. This paper describes and explores the Lagrange mean value theorem-based method to Analysis of Finite Fluctuations. The sole restriction on the suggested method is that the model has to be differentiable. The suggested approach's consistency is demonstrated by contrasting it with the computation of School indices. An investigation is conducted into the procedure's numerical stability. In contrast to current methods, our methodology avoids using the original model's approximation, which prevents inaccurate findings from being obtained.

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