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CONCEPTS OF SOME TOPOLOGICAL LINEAR SPACES

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Abstract

The study of sequence spaces has vastly been enriched in many different directions by a number of Mathematicians who have made their significant contributions to it with an abundance of new and fruitful ideas. Attempts have also been made to generalize the theory of scalar valued sequence spaces to vector valued sequence spaces. In the present thesis we have made our efforts in this direction which generalizes and unifies the study of various existing sequence spaces constructed by earliar workers and also gives a number of ineresting properties of many other such spaces. In this artilce we have given list of various sequence spaces constructed earliar by many workers. We have also given definitions and some important results about K¨othe-Toeplitz duals, Orlicz Sequence Spaces and Bilateral sequences and series etc. which have been widely studied.

Keywords: Topological, Spaces, Linear, Generalized etc.

1. INTRODUCTION

Functional Analysis grew up in the nineteenth century. It refers to the studies where the essence is the use of abstract methods [1]. These abstract methods consist of applying theorems about functions between sets having an algebraic and a topological or more general a limit structure. These structured sets are called spaces and the functions between them are called functionals. Linear functional analysis refers to the part of the discipline where only linear functionals are involved [2].Literatures concerning this can be found in standard text books of Functional Analysis. This article is of introductory nature.

2. LIST OF SEQUENCE SPACES

We shall denote

N = Set of all natural numbers, R = Set of all real numbers

C = Set of all complex numbers, Z = Set of all integers

Z^+ = Set of all positive integers, Z^- = Set of all negative integers

ω = Class of all complex valued sequences

Following are some of the important classes of sequences and functions, in what $\bar{a}= (a_k)$ we shall mean $\bar{a}= (a)$ ∞ 1. Similarly we can write $\bar{p}= (p_k)$, $\bar{q}= (q_k)$, $\bar{\lambda}= (\lambda_k)$ and $\bar{\mu}= (\mu_k)$ where $\bar{p}$ and $\bar{q}$denote sequences of strictly positive real numbers and $\bar{\lambda} = (\lambda_k)$, $\bar{\mu} = (\mu_k)$ be sequences of non zero complex numbers [3].

1. $c_o(p) = \{\bar{a} = (a_k) \in \omega : |a_k|^{pk} \to 0 \text{ as } k \to \infty\}$

2. $c(p) = \{\bar{a}= (a_k) \in \omega : \exists\, l \in C \text{ such that } |a_k - l|^{pk} \to 0 \text{ as } k \to \infty\}$

3. $l_\infty(p) = \{\bar{a}= (a_k) \in \omega : \sup_k|a_k|^{pk} < \infty\}$

4. $$\ell(p) = \left\{\bar{a} = (a_k) \in \omega : \sum_{1}^{\infty} |a_k|^{p_k} < \infty\right\}$$

where $c_o(p)$, $c(p)$, $l_\infty(p)$, $l(p)$ appear in other works [4].

If X denotes the normed space and x_k's are the elements of X then

5. $c_o(X) = \{\bar{x} = (x_k) \in X : \|x_k\| \to 0 \text{ as } k \to \infty\}$

6. $c(X) = \{\bar{x} = (x_k) \in X : \exists\, l \in X \text{ such that } \|x_k - l\| \to 0 \text{ as } k \to \infty\}$

7. $l_\infty(X) = \{ \bar{x} = (x_k) \in X : \sup_k\|x_k\| < \infty\}$

8. $$\ell_p(X) = \left\{ \bar{x} = (x_k) \in X : \sum_1^\infty \|x_k\|^p < \infty \right\} ; 0 < p < \infty.$$

$c_o(X)$, $c(X)$, $l_\infty(X)$, $l_p(X)$ are used by others [5].

9. $c_o(X, \lambda, p) = \{ \bar{x} = (x_k) \in X : \|\lambda_k x_k\|^{pk} \to 0 \text{ as } k \to \infty\}$

10. $c(X, \lambda, p) = \{ \bar{x} = (x_k) \in X : \exists\, l \in X \text{ such that } \|\lambda_k x_k - l\|^{pk} \to 0 \text{ as } k \to \infty\}$

11. $l_\infty(X, \lambda, p) = \{ \bar{x} = (x_k) \in X : \sup_k \|\lambda_k x_k\|^{pk} < \infty\}$

12. $$\ell(X, \lambda, p) = \left\{ \bar{x} = (x_k) \in X : \sum_1^\infty \|\lambda_k x_k\|^{p_k} < \infty \right\} , \ 0 < p < \infty$$

$c_o(X, \lambda, p)$, $c(X, \lambda, p)$, $l_\infty(X, \lambda, p)$, $l(X, \lambda, p)$ were introduced and investigated and have been further developed by many others [6].

3. KOTHE-TOEPLITZ DUALS

In classical theory of matrix transformations, one of the basic problems is the characterization of matrices which map a sequence space E into a sequence space. These duals known as Ko¨the-Toeplitz duals or $\bar{\alpha}$and $\bar{\beta}$duals are defined as follows:

Definition 3.1: The β− dual of $E \subset \omega$ is defined as

$$E^\beta = \left\{ \bar{a} = (a_k) \in \omega : \sum_1^\infty a_k x_k \ converges\ for\ all\ \bar{x} = (x_k) \in E \right\}$$

where ω denote the space of all sequences of complex numbers and $E \subset \omega$

Definition 3.2: The $\bar{\alpha}$dual of E ⊂ ω is defined as

$$E^{\alpha} = \left\{ \bar{a} = (a_k) \in \omega : \sum_{1}^{\infty} |a_k x_k| < \infty, for\ all\ \bar{x} = (x_k) \in E \right\}$$

For sequence spaces other kinds of duals such as δ−dual and γ−dual etc. have also been investigated [8].

Theorem 3.1: For any $p = (p_k)$ and $p_k > 1$, for every k, then $l^{\beta}(p) = M(p)$, where

$$M(p) = \bigcup_{N>1} \left\{ \bar{a} = (a_k) \in \omega : \sum_{1}^{\infty} |a_k|^{q_k} N^{-q_k} < \infty \right\}, \text{ with } p_k^{-1} + q_k^{-1} = 1$$

Theorem 3.2:

(a)If $0 < p_k \leq 1$, for all k then $l^*(p)$ is isomorphic to $l_{\infty}(p)$;

(b) If $1 < p_k \leq \sup_k p_k < \infty$, for all k, then $l^*(p)$ is isomorphic to M(p);

(c) If $1 < \inf_k p_k \leq \sup_k p_k \leq \infty$ and l(p) has its natural paranorm topology then $l^*(p)$ is linearly homeomorphic to l(p) [7].

4. GENERALIZED KOTHE- TOEPLITZ DUALS

During the course of generalization of complex sequence space theory to vector valued space theory, i.e., when complex sequences were replaced by vector sequences and characterizaion of matrices of operators instead of complex numbers, the first step is to determine the generalized K¨othe-Toeplitz duals, also known as the generalized α− and β− duals which are defined as follows:

Let X and Y be Banach spaces and B(X, Y) denote the Banach space of bounded linear operators from X into Y .

Definition 4.1: LetA^{-} = (A_k), a sequence of linear, but not necessarily bounded, operators A_k on X into Y. Suppose E(X) is a non-empty set of $\bar{X}$ valued sequences. Then the α− dual of E(X) is defined by

$$E^{\alpha}(X) = \left\{ \bar{A} = (A_k) : \sum_{1}^{\infty} ||A_k x_k|| \; converges \; for \; all \bar{x} = (x_k) \in E(X) \right\}$$

and β− dual of E(X) is defined as

$$E^{\beta}(X) = \left\{ \bar{A} = (A_k) : \sum_{1}^{\infty} A_k x_k \; converges \; in \; the \; norm \; of \; Y, \forall \; \bar{x} = (x_k) \in E(X) \right\}$$

The following concept was introduced and was later on termed as 'group norm', for characterization of $\bar{\beta}$duals of certain sequence spaces [8].

Definition 4.2: Let (A_k) = (A_1, A_2, A_3, ...) be a sequence in B(X, Y). Then the group norm of (A_k) is defined as

$$||(A_k)|| = \sup \; ||\sum_{1}^{n} A_k x_k||$$

where the supremum is taken over all n ∈ N and all xk∈ S, (S is the closed unit sphere in X).

The group norm may or may not be finite, though we are usally concerned with problems which give rise to finite group norms. Some elementary properties of group norm are as follows [9]:

Theorem 4.1: Let (A_k) be a sequence in B(X, Y) and denote R_n = (A_n, A_{n+1}, A_{n+2}, ...). Then

(a) $||A_m|| \leq ||R_n||$, for all $m \geq n$;

(b) $||R_{n+1}|| \leq ||R_n||$, for all $n \in N$;

(c) $$||\sum_{n}^{n+p} A_k x_k|| \leq ||R_n||.max\{ ||x_k|| : n \leq k \leq n+p\}$$

for any x_k and all $n \in N$ and all non-negative integers p.

Initially, to obtain the operator version of the classical results all the operators were taken bounded, but later on this restriction was removed and the cases where all but finite number of operators were bounded attracted the attention of the workers. We are giving here some of the characterizations [10].

Theorem 4.2: $(A_k) \in c_o^{\alpha}(X, \lambda, p)$ if and only if

(a) there exists $m \in N$ such that $A_k \in B(X, Y)$, for all $k \geq m$;

(b) $$\sum_{m}^{\infty} ||A_k|| \leq \infty$$

Connected with the group norm following lemma is proved:

Lemma 4.1: Let (T_k) be a sequence in B(X, Y) and denote

$R_k = (T_k, T_{k+1}, T_{k+2}, ...)$. Then exactly one of the following is true [11],

(a) $||R_k|| = \infty$, for all $k \geq 1$;

(b) $||R_k|| < \infty$, for all $k \geq 1$.

In the above theory, operator versions of K¨othe-Toeplitz duals have been obtained for vector valued sequence spaces by taking sequence $\bar{x}$= (x_k) in Banach space X and members of K¨othe-Toeplitz duals as sequence $\bar{A}$ = (A_k) of operators whether all bounded or not. In a sense the study of converse of this problem has also been investigated when space of sequences $\bar{A}$= (A_k) are taken with $A_k \in$ B(X, Y) and members of Kothe-Toeplitz duals $\bar{x}$= (x_k) are taken in the Banach space X. By making use of Hahn-Banach Theorem $\bar{\alpha}$and $\bar{\beta}$duals of various spaces of B(X, Y) -valued sequences have been determined [12].

5. ORLICZ FUNCTION AND ORLICZ SEQUENCE SPACES

The study of Orlicz sequence spaces was initiated with a certain specific purpose in Banach space theory. It got interested in Orlicz spaces in connection with finding Banach spaces with symmetric Schauder bases having complementary subspaces isomorphic to c◦ or ` p , 1 ≤ p < ∞ . Subsequently Lindenstrauss and Tzafriri studied important and interesting structural problems in Banach spaces. Orlicz spaces find a number of useful applications in the theory of non-linear integral equations. Orlicz sequence spaces are the special case of Orlicz spaces.

The definition of Orlicz function and Orlicz sequence spaces are as follows:

Definition 5.1: An Orlicz function M: [0,∞) → [0,∞) is a continuous, non-decreasing and convex function defined for t ≥ 0 such that

(i) M(x) > 0 for x > 0;

(ii) M(0) = 0 and

(iii) $\lim_{t\to\infty} M(t) = \infty$

An Orlicz function M can always be represented in the following integral form.

$$M(x) = \int_0^x p(t)dt$$

where p is known as the kernal of M, is right differentiable for $t \geq 0$, $p(0) = 0$, $p(t) > 0$ for $t > 0$, p is non-decreasing and $p(t) \to \infty$ as $t \to \infty$.

Definition 5.2: Lindenstrauss and Tzafriri used the ideas of Orlicz function to construct the sequence s_p

$$l_M = \left\{ x \in \omega : \sum_1^\infty M\left(\frac{|x_k|}{\rho}\right) < \infty, \ for\ some\ \rho > 0 \right\}$$

The space l_M with the norm

$$||x|| = \inf\left\{ \rho > 0 : \sum_1^\infty M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}$$

6. BILATERAL SEQUENCES AND SERIES

Definition 6.1: A bilateral sequence in a set S is a function $\bar{a}$ on Z to S. We generally denote it by $\bar{a} = (\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots)$ or $\bar{a} = (a_k)_{-\infty}^{\infty}$ where $a(n) = a_n \in S$ for $n \in Z$.

Definition 6.2: We define a bilateral series by an expression $\sum_{-\infty}^{\infty} a_k$ where a_k's are real or complex numbers.

Definition 6.3: A bilateral series $\sum_{-\infty}^{\infty} a_k$ is said to converge absolutely if each series $\sum_0^{-\infty} a_k$ and $\sum_1^{\infty} a_k$ converge absolutely.

It is well known that these bilateral sequences and series appear and play a vital role in the theory of real or complex analysis, in the representation of functions such as Laurent series expansion, Fourier series expansion and Hypergeometric series expansion of functions etc.

7. CONCLUSION

It is concluded that the study of topological linear spaces is in turn a branch of Functional Analysis in which sequence spaces have widely been studied. A sequence space is the special case of the more general study of Function space. In particular, a sequence space is an abstract space of infinite dimension. The historical roots of Functional Analysis lie in the study of these spaces and was mainly concerned with the problems in Fourier series, Power series and system of equations with infinitely many variables.

REFERENCES

[1]. C. Park and C. Alaca, An introduction to 2-fuzzy n-normed linear spaces and a new perspective to the Mazur-Ulam problem, J. Inequal. Appl. 14, 2012.

[2]. H. Z. Ibrahim, α−topological vector spaces, Sci. J. Uni. Zakho. 5 (1) (2017), 107-111.

[3]. M. D. Khan, S. Azam and M.S. Bosan, s-topological vector spaces, J. Linear. Topological. Algebra. 04(02) (2015), 153-158.

[4]. M. D. Khan, S. Azam, M. S. Bosan, s-topological vector spaces, J. Linear. Topological. Algebra. 4 (2) (2015), 153-158.

[5]. M. Khan, M. I. Iqbal, On irresolute topological vector spaces, Adv. Pure. Math. 6 (2016), 105-112.

[6]. M. Mursaleen and Q.M. Danish Lohani, Intuitionistic fuzzy 2-normed space and some related concepts, Chaos SolitonsFract. 42 (1), 224–234, 2009

[7]. M. Radhi, J. H. Hussein, S. K. Hameed ,On Semi - Pre irresolute Topological Vector Space, Diy. ofpur. Sci V.14 No.3 pp183-191, 2017.

[8]. M.H.M. Rashid and Lj.D.R. Kočinac, Ideal convergence in 2-fuzzy 2-normed spaces, Hacet. J. Math. Stat. 46 (1), 145–159, 2017.

[9]. N. Rajesh, V. Tanjavur, , On Strongly Preirresolute Topological Vector Space, MathematicaBohemica, Vol. 138, No.1, pp 37-42, 2013

[10]. R.M. Somasundaram and T. Beaula, Some aspects of 2-fuzzy 2-normed linear spaces, Bull. Malays. Math. Sci. Soc. 32 (2), 211–221, 2009.

[11]. T. Bag and S.K. Samanta, Finite dimensional intuitionistic fuzzy normed linear spaces, Ann. Fuzzy Math. Inform. 6 (2), 45–57, 2013.

[12]. Y. A. Abou-Elwan, Some properties of β−continuous mappings, β−open mappings and β−homeomorphism, Middle-East J. of Sci. Res. 19 (12) (2014), 1722-1728.