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The Governing Equations' Function in Fluid Dynamics

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Abstract: The symmetrical building block of numerous equations of motion that explain the flow in many areas is fluid dynamics. Most fluid dynamics issues can be resolved by measuring variables such as velocity, wall shear stress, resistivity, and other characteristics that depend on time and space coordinates. In this essay, we attempt to describe the various fluid dynamics equations of motion.

Governing Equations for Ideal fluid flow:

(i) Equation of Continuity :- The equation of continuity is mass balance equation based on the law of conservation of mass.

If M be the mass of fluid contained within the volume V at time t, then

$$M = \int_V \rho dV \qquad \ldots(1)$$

Where, ρ is density of the fluid.

The mass flux through the surface S enclosing volume V

$$= \int_S \rho \vec{q} \,.\, dS \qquad \ldots(2)$$

Then by mass balance equation

$$\frac{d}{dt}\int_V \rho dV = -\int_S \rho \vec{q}.\, dS \qquad \ldots(3)$$

with the use of Divergence theorem and vector identity for divergence, above equation reduces to

$$\frac{D\rho}{Dt} + \rho \text{ div } \vec{q} = 0 \qquad \ldots(4)$$

where $\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{q}\,.\Delta$ is material derivative.

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for steady flow, $\frac{\partial}{\partial t}(\,.\,) = 0$, thus

$$\text{div}\,(\rho\,\bar{q}) = 0 \qquad \ldots(5)$$

For incompressible flow i.e. density of the fluid remains constant. Then

$$\text{div}\,\vec{q} = 0 \qquad \ldots(6)$$

(ii) Equation of motion (Euler Equation)

It is based on Newton's second law of motion which states -

"The rate of change of momentum of a specified region of the fluid is equal to the vector sum of all forces which act on that region of the fluid". Here, force is stress, which is the pressure directed along the inward normal at all points and given by

$$\text{Stress} - -\int_{\Sigma} pn\,d\Sigma \qquad \ldots(7)$$

$$\text{body force} = \int_{\tau} \rho F\,d\tau \qquad \ldots(8)$$

where Σ is the closed surface bounded the volume τ and F be the body force per unit mass. Also,

$$\text{rate of change of momentum} = \frac{d}{dt}\int_{\tau} \rho\bar{q}\,d\tau \qquad \ldots(9)$$

where $\bar{q} = u\hat{i} + v\hat{j} + w\hat{k}$ is velocity vector. Then by Newton's second law.

$$\frac{d}{dt}\int_{\tau} \rho\,\bar{q}\,d\tau = -\int_{\Sigma} pn\,d\Sigma + \int_{\tau} \rho F d\tau \qquad \ldots(10)$$

As $$\frac{d}{dt}\int_{\tau} \rho u\,d\tau = \int_{\tau} \left[\frac{D}{Dt}(\rho u) + \rho u\,\text{div}\,\bar{q}\right] d\tau$$

$$= \int_{\tau} \left\{\rho\frac{Du}{dt} + u\left[\frac{D\rho}{Dt} + \rho\,\text{div}\,\bar{q}\right]\right\} d\tau \qquad \ldots(11)$$

On writing similar results for v and w and summing. Then in view of the continuity equation

$$\frac{D\rho}{Dt} + \rho\,\text{div}\,\bar{q} = 0 \qquad \ldots(12)$$

Equation (1.11.10) becomes :

$$\int_{\tau} \rho \frac{D\bar{q}}{Dt} d\tau = -\int_{\Sigma} \rho n \, d\Sigma + \int_{\tau} \rho \, \vec{F} d\tau \qquad \ldots(13)$$

With the use of Gauss Divergence Theorem and the equation $\frac{D}{Dt} = \frac{\partial}{\partial t} + \bar{q}.\nabla$, above equation reduces into

$$\int_{\tau} \left(\rho \frac{D\vec{q}}{Dt} + \nabla p - \rho \vec{F} \right) d\tau = 0 \qquad \ldots(14)$$

But τ is arbitrary, thus at any point of the fluid

$$\frac{D\bar{q}}{Dt} = \vec{F} - \frac{1}{\rho} \nabla p \qquad \ldots(15)$$

Equation (15) is known as Euler's equation of motion for an ideal fluid.

Governing Equations for Viscous Fluid Flow

(i) Equation of continuity (conservation of mass) :- It amounts to the basic physical law, that is, the matter is conserved, it is neither being created nor destroyed.

Mass of fluid entering per unit time, in the controlled surface S is

$$= -\int_{S} \rho v_j n_j \, dS = -\int_{V} \frac{\partial}{\partial x_j} (\rho v_j) \, dV \qquad \ldots(16)$$

n_j is unit outward normal to the surface dS

The rate at which the enclosed mass increases

$$= \frac{\partial}{\partial t} \int_{V} \rho \, dV$$

$$= \int_{V} \frac{\partial \rho}{\partial t} dV \qquad \ldots(17)$$

By Mass Balance equation.

$$\int_{V} \frac{\partial \rho}{\partial t} dV = -\int_{V} \frac{\partial}{\partial x_j} (\rho \, v_j) \, dV$$

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or $$\int_V \left\{ \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho v_j) \right\} dV = 0$$

Since V is arbitrary chosen volume, we deduce that

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho v_j) = 0 \qquad \ldots(18)$$

which is the required equation of continuity in tensor notations.

In vector notation, it is

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \vec{v}) = 0 \qquad \ldots(19)$$

In cylindrical co-ordinates (r, θ, z) the equation of continuity is

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r \rho v_1) + \frac{\partial}{\partial \theta}(\rho v_2) + \frac{\partial}{\partial z}(r \rho v_3) \right] = 0 \qquad \ldots(20)$$

(ii) Equations of motion (Navier-Stokes equation): conservation of momentum- The equation of motion are derived from Newton's second law which states that

Rate of change of linear momentum = Total force …(21)

Rate at which the momentum increases in the enclosed volume V

$$= \frac{\partial}{\partial t} \int_V \rho \, v_i \, dV \qquad \ldots(22)$$

Rate of out flow of momentum through the controlled surface S

$$= \int_S v_i (\rho v_j \, n_j) dS \qquad \ldots(23)$$

Body forces acting on the enclosed volume V

$$= \int_V \rho \, f_i \, dV \qquad \ldots(24)$$

and surface forces acting on the controlled surface S

$$= \int_S p_i \, dS \qquad \ldots(25)$$

Then by equation (1.12.6), we have

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$$\frac{\partial}{\partial t}\int_V \rho V_i \, dV + \int_S v_i(\rho v_j n_j) dS = \int_V \rho f_i \, dV + \int_S p_i \, dS \qquad \ldots(26)$$

where $f_i \rightarrow$ body force per unit mass

$p_i \rightarrow$ force on the boundary per unit area.

$$p_i = \sigma_{ij} n_j \qquad \ldots(27)$$

$v_i \rightarrow i^{th}$ component of velocity

and

$$\sigma_{ij} = -p\,\delta_{ij} + \tau_{ij} \qquad \ldots(28)$$

σ_{ij} is the stress tensor acting on an area perpendicular to the x_i-axis, taken in the direction parallel to x_j-axis.

τ_{ij} is the shear stress exerted on an fluid surfaceof constant x_i by the fluid in lower region than x_i.

Substituting equations (12) and (13) in the equation (11) and using Gauss Divergence theorem and also noting that V is an arbitrary chosen volume, we get the equation of motion as

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}(\rho v_i v_j) = \rho f_i - \frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} \qquad \ldots(29)$$

Using the equation of continuity, we get

$$\rho\left[\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j}\right] = \rho f_i - \frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} \qquad \ldots(30)$$

For isotropic Newtonian fluid, the constitutive equation

$$\tau_{ij} = 2\mu\, e_{ij} - \frac{2}{3}\mu\, e_{kk}\,\delta_{ij} \qquad \ldots(31)$$

where $e_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$ $\qquad \ldots(32)$

Using these equations , we obtain

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$$\rho\left[\frac{\partial v_i}{\partial t}+v_j\frac{\partial v_i}{\partial x_j}\right]=\rho f_i-\frac{\partial p}{\partial x_i}+\frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}-\frac{2}{3}\delta_{ij}\frac{\partial v_k}{\partial x_k}\right)\right] \quad \ldots(33)$$

These are known as Navier-Stokes equation for the motion of a viscous compressible fluid.

Governing equation for MHD flow Maxwell equation : These are the equations, formulated by James clerk Maxwell that gives the complete description of the production and interrelation of electric and magnetic fields. These equations are :

(i) $\nabla.\ \vec{E}=\frac{\rho}{\epsilon_0}$

(ii) $\nabla.\ \vec{B}=0$

(iii) $\nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$

(iv) $\nabla\times\vec{H}=\vec{J}+\frac{\partial\vec{D}}{\partial t}$

where

$\vec{E}\rightarrow$ electric field

$\vec{B}\rightarrow$ magnetic flux density

$\rho\rightarrow$ free electric charge density

$\epsilon_0\rightarrow$ electric permeability in free space

$\vec{J}\rightarrow$ free current density

$\vec{D}\rightarrow$ electric displacement field

$\vec{H}\rightarrow$ magnetic field.

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