International Journal of Research in Engineering and Applied Sciences(IJREAS)
Available online at http://euroasiapub.org/journals.php
Vol. 12 Issue 11, November- 2022
ISSN(O): 2249-3905, ISSN(P) : 2349-6525 | Impact Factor: 8.202 |
Thomson Reuters ID: L-5236-2015

Embedding Theorem for the Space of S-convex Sets

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Abstract

In this present note, our focus has been on the in-depth exploration of S-convex sets, a concept with profound mathematical implications. We have not only introduced the idea of S-convex sets but have gone a step further to define algebraic operations that can be performed on the set containing all such S-convex sets. What makes this endeavor particularly intriguing is the revelation that these algebraic operations consistently yield results that remain within the realm of S-convex sets, thereby establishing a strong foundation for the mathematical framework we have developed. One of the key highlights of our work is the utilization of this structure to prove an embedding theorem for the spaces of S-convex sets.

Keywords: Convex set, generalized convex set, normed space.

1. Introduction

Convex sets are fundamental constructs in mathematics and have far-reaching applications in various disciplines, including optimization, economics, engineering, and computer science. A convex set is a set that exhibits a particular geometric property, and understanding this property is essential for a wide range of theoretical and practical problems. This introduction provides an overview of convex sets and offers a concise literature survey to highlight their importance and applications.

A convex set in Euclidean space is a set that contains all line segments connecting any two points within the set. More formally, a set S is convex if, for every pair of points x and y in S, the entire line segment between x

and y lies entirely within S. This geometric definition of convexity forms the basis for a vast field of mathematical theory and practical applications.

Convex sets have been studied extensively, leading to a rich body of literature. Researchers have explored their properties, characterizations, and applications across various domains. We can see its applications in different areas. In [4], the author introduces a discrete framework for approximating solutions to 2D shape optimization problems under convexity constraints, using the support function or gauge function, ensuring discrete convex shapes and ease of implementation with diverse objective functions and width/diameter constraints.

Understanding convex sets is pivotal as they serve as a fundamental building block for convex optimization, which, in turn, is a powerful tool in solving various real-world problems. In [12] the extreme points and related results. Convex sets underpin the efficient algorithms used in resource allocation, portfolio optimization, image reconstruction, and many other fields. Their geometric properties ensure that optimization problems associated with convex sets have unique solutions, making them an indispensable concept in mathematical modeling.In [9], the authors introduce a convex-constrained image restoration model that effectively improves image quality by optimizing the relaxation. We can see the literature regarding the properties of convex sets in [1-3, 6-8, 10, 11, 13] and references therein and their applications.

In the ensuing paper, we will delve deeper into the properties of S-convex sets and its algebraic structure. In the next section, we give the definitions and define Minkowski sum and scalar product for these sets. We also study its behavior that they are S-convex sets. In the section 3, we give embedding theorem for S-convex sets.

2. Basic definitions

Definition: (Convex set) Let A be a non-empty subset of a linear space E now for $x, y \in A$ and $\lambda, \mu \geq 0$, then$A$ is called a Convex Set whenever,

$\lambda x + \mu y \in A$ for $\lambda + \mu = 1$.

Definition: (S-Convex set) Let A be a non-empty subset of a linear space E. Now, for every x, $y \in A$ and $\lambda, \mu \geq 0$ b (scalars) we shall say that A is a S-Convex Set if, $\lambda x + \mu y \in A$ for $\lambda+\mu \leq 1$.

Thus, every S-Convex Set is inherently a Convex Set, but the reverse statement does not necessarily hold. This insight underscores the notion that an S-Convex Set represents a broader category than Convex Sets. Put differently, a Convex Set can be seen as a specific instance or a special case within the broader category of S-Convex Sets.

Operations on S -convex sets:

Definition: (Minkowski sum) Let A, B are two non-empty S-convex sets. A Minkowski sum of these two sets is defined as:

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$$A + B = \{x: x = a + b, a \in A, b \in B\}.$$

It can be easily observed that Minkowski sums of two S-convex sets is again a S-convex set.

Definition: (Scalar product) Let A be any non-empty S-convex set and $\alpha \in R$ be any scalar. Define the scalar product as:

$$\alpha A = \{x: x = \alpha a, a \in A\}.$$

Theorem: Suppose a set $A \subset R^n$ is S-convex set, then the set $T = \{x_i: P(x) = x_i, x = (x_1, x_2, \dots, x_n) \in A\}$ under the projection map P is also S-convex set.

Proof: Consider A is an S-convex set. We need to prove that T is also an S-convex set.

Let $x_i, y_i \in T$, then there exists $x, y \in A$ two elements such that,

$P(x) = x_i, P(y) = y_i$. Using the S-convexity of A, we can say that,

$\lambda x + \mu y \in A,$ for $0 \leq \lambda + \mu \leq 1$.

Now, by projection, we get $P(\lambda x + \mu y) = \lambda x_i + \mu y_i$.

By which, we can write $\lambda x_i + \mu y_i \in T$.

Hence, T is S-convex set.

Theorem: Suppose $A \subset R^n$, $B \subset R^m$ are S-convex sets. Prove that,

$$A \times B = \{(x_1, x_2): x_1 \in A, x_2 \in B\}$$

is also S-convex set.

Proof: Let $(x_1, x_2), (y_1, y_2) \in A \times B$.

We get $x_1, y_1 \in A$, and A is an S-convex set, which gives for $0 \leq \lambda + \mu \leq 1$

$$\lambda x_1 + \mu y_1 \in A.$$

Similarly, $x_2, y_2 \in B$, using the S-convexity of B and $0 \leq \lambda + \mu \leq 1$gives

$$\lambda x_2 + \mu y_2 \in B.$$

Now, we get $(\lambda x_1 + \mu y_1, \lambda x_2 + \mu y_2) \in A \times B$.

Which gives $A \times B$ is an S-convex set.

Theorem: Let A be any S-convex set. Prove that αA is lso S-convex set.

Proof: It can be proved easily, so we left it to the reader.

3. Main Results

Now, we define the algebraic structure for the collection of S-convex sets.

Let M be a set whose elements are S-convex sets.Let A and B are two S-convex sets and α is

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any real number, defineMinkowski sum and scalar product as defined above.

It can be seen easily that,

A+(B+C)=(A+B)+C,

A+B=B+A,

By these properties, we can say that the set M consisting of all S-convex sets is a commutative semi-group under addition.

Theorem 1. Let M be a commutative semigroup of S-convex sets in which the cancellation law holds. That is for any A, B, C ∈M, we have the following properties:

i) (A+B)+C=A+ (B+C)

ii) A+B=B+A

iii) A+C=B+C implies A=B.

Then, M can be embedded in a group N. Further N can be chosen to be minimal in the following sense:

If G is any group in which M is embedded, then N is isomorphic to a subgroup of G containing M.

Proof: Firstly, we define elements for N. It consists of equivalence classes of pairs (A, B) of elements of M. For this, we define the equivalence relation denoted by '~' as

$$(A,B)\sim(C,D)$$

iff $A+D=B+C$.

It is easy to check this is an equivalence relation. The equivalence class containing the pair (A, B) is denoted by [A, B].

For any element A in M, we identify [A+B, B] in N.

Let define operations on N.

For $[A,B],[C,D]\in N$,

$$[A,B]+[C,D]=[A+C,B+D].$$

For any non-negative scalar λ, [A, B] ∈ N,

$\lambda\,[A,B]=[\lambda A,\lambda B]$.

If $\lambda<0$, $\lambda\,[A,B]=[|\lambda|A,|\lambda|B]$.

The metric on N is defined as:

$$\partial([A,B],[C,D])=d(A+D,B+C).$$

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Now, we prove the uniqueness of functions defined above.

Let $[A, B] = [A_1, B_1]$ and $[C, D] = [C_1, D_1]$.

$$\partial([A, B], [C, D]) = d(A + D, B + C),$$

add $A_1 + B_1 + C_1 + D_1$ and get:

$$\begin{aligned}\partial([A, B], [C, D]) &= d(A + D, B + C)\\ &= d(A + D + A_1 + B_1 + C_1 + D_1, B + C + A_1 + B_1 + C_1 + D_1)\\ &= d(A_1 + D_1, B_1 + C_1)\end{aligned}$$

$= \partial([A_1, B_1], [C_1, D_1])$.

Now, we can easily prove by using Graves [5].

Theorem 2. If a multiplication by a non-negative real scalar satisfying

i) $\lambda(A+B)= \lambda.A+ \lambda.B$

ii) $(\lambda_1+ \lambda_2)A= \lambda_1 A+ \lambda_2 A$

iii) $\lambda_1(\lambda_2 A)= \lambda_1 \lambda_2 A$

iv) 1.A=A,

then a multiplication by real scalars can be defined on N so as to make N a vector space and so that for $\lambda \geq 0$ andA $\in$ M, the product λA coincides with the one given in M.

Proof: The proof for these is straightforward but lengthy. We left it to the reader to prove.

Theorem 3. If we define a metric on M that satisfy:

i) D(A+C,B+C)=d(A,B)

ii) D(λA, λB)= λd(A,B)

iii) A+B and λA are continuous operations in the topology defined on d, then a metric can be defined on N to make N a normed linear space and so that if A, B $\in$M, the distance between A and B equals d(A,B).

Proof: We must prove that ∂, defined as in above theorem, is metric.

Consider $x, y \in N$ $such$ $that$ $x = [A, B], y = [C, D]$.

Suppose $\partial(x, y) = 0$. It gives $d(A + D, B + C) = 0$. Since d is a metric, thus we get:

$$A + D = B + C$$

$$\Rightarrow (A, B) \sim (C, D)$$

$$\Rightarrow x = y.$$

For $x = y$, it is easy to prove that $\partial(x, y) = 0$.

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By using the symmetry of d metric, we can easily see $\partial(x,y) = \partial(y,x)$.

Triangle inequality: Let $z = [E,F]$.

Thus, $\partial(x,z) = d(A+F, B+E) = d(A+D+C+F, B+C+D+E) \leq d(A+D, B+$ $C+dC+F, D+E=\partial x, y+\partial(y, z)$.

Now, ∂ is a function of one variable as it is invariant translation because,

$$x - y = x + (-1)y.$$

Thus, define $d(x,y) = \|x - y\|$.It will satisfy all the properties of norm. Hence, the proof.

Now, we prove the cancellation law for certain classes of S-convex sets.

Lemma 1. Let A, B, and X be given in areal normed linear space. Suppose B is closed and S-convex, X bounded, and

$$A + X \subset B + X,$$

then, $A \subset B$.

Proof: Let $a \in A$ be any element. We will show $a \in B$.

For any $x_1 \in X$, we get $a + x_1 \in A + X$. Since $A + X \subset B + X$. It gives $a + x_1 \in B + X$ that is there exists some $b_1 \in B$ and $x_2 \in X$ such that$a + x_1 = b_1 + x_2$. Repeating the above process again, we get$b_2 \in B$ and $x_3 \in X$ such that $a + x_2 = b_2 + x_3$.

Repeating it infinitely and taking summation over first n equations, we get:

$$na + \sum_{j=1}^{n} x_j = \sum_{j=1}^{n} b_j + \sum_{j=1}^{n} x_{j+1}.$$

By some simple algebraic calculations, we get:

$$a = \frac{1}{n}\sum_{j=1}^{n} b_j + \frac{x_{n+1}}{n} - \frac{x_1}{n}.$$

With the help of S-convexity and closedness of B, we can say that $b \in B$. Hence, the result.

Lemma 2. If A, B are closed, S-convex sets in a real normed linear space and X is bounded, then $A + X = B + X$ implies $A = B$.

Lemma 3. Let A and B are closed, S-convex sets in a real normed linear space. Suppose also that A+ λT and B+ λT are closed for all λ ≥ 0, where T is the unit sphere. Let X be any bounded set. Then,

$$d(A,B) = d(A+X, B+X).$$

Proof: Let S is the unit sphere of the space. Consider the following inequalities:

i) B⊂A+ λS

ii) A⊂ B+λS

iii) B+X⊂ A+X+λS

iv) A+X⊂ B+X+λS.

Let $d_1 = d(A, B)$ and $d_2 = d(A + X, B + X)$.

We can observe that $d_1$is the infimum of all the positive numbers for which i) and ii) hold. Similarly, $d_2$is the infimum for which iii) and iv) hold. We can obtain iii) and iv) from i), ii) by adding X on both sides, thus $d_2 \leq d_1$.

Conversely, by Lemma 1, we can say $d_1 \leq d_2$. Thus $d_1 = d_2$.

Hence, the proof.

With the help of all the proved results, we are in the situation to prove the embedding theorem.

Theorem 4. Let P be a real normed linear space and Q be any space having the points which are closed, bounded S-convex sets in P, and which has the following properties:

i) Q is closed under addition and multiplication by a non-negative scalar.

ii) If A∈Q and T is the unit sphere of P, then A+T is closed.

iii) P is motorized by the Hausdorff metric.

Then Q can be embedded as a S-convex cone in a real normed linear space R in such a way that

a. the embedding is isometric.
b. addition in R induces addition in Q.
c. multiplication by non-negative scalars in Rinduces the corresponding operation in Q.

Furthermore, R can be chosen to be minimal in the following sense:

If I is any real normed linear space in which Q is embedded in the above fashion, then I contains a subspace containing Q and isomorphic to R.

Proof: We can verify the conditions 1-10 by using Theorem 1 except 3 and 8, which can be seen from Lemma 2 and Lemma 3. Thus, the proof.

Conclusion: In conclusion, this note has provided a comprehensive exploration of S-convex

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sets, introduced the concept, and defined algebraic operations on the set containing all S-convex sets. Importantly, we have demonstrated that these algebraic operations yield results that remain within the realm of S-convex sets,

underscoring the robustness and self-consistency of this mathematical framework.

Furthermore, the utility of these operations has been leveraged to establish an embedding theorem for the spaces of S-convex sets. This result not only deepens our understanding of S-convexity but also extends its applicability to a broader mathematical context. The embedding theorem offers new avenues for research and applications, highlighting the significance of S-convex sets in various mathematical disciplines.

This note serves as a valuable contribution to the study of S-convex sets, opening possibilities for further exploration and practical applications in mathematics.

and related fields.

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