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Common Fixed Point Theorem in Complete G-Metric space for Six Maps

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Abstract: In this paper a common fixed point theorem for six maps in complete G-metric space is proved for integral type contractive conditions using continuity, weak commuting and weakly compatibility.

Keywords: Weakly Compatible maps, Contractive condition, G-Metric space.

1.Introduction and preliminaries

Mustafa and Simsintroduced a new generalised G-metric spaces as a generalization of metric spaceas follows.

Definition 1.1 [75] "Let $G: X \times X \times X \to R^+$ be a function on a non-empty X satisfying

(G-1) $G(x, y, z) = 0$ if $x = y = z$,

(G-2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,

(G-3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$,

(G-4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$ (symmetry in all three variables),

(G-5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$, (rectangle inequality).

The function G is called a generalized metric or more specifically, a G-metric on Xand the pair (X, G) is called a G-metric space."

Definition 1.2 [75] "A sequence $\{x_n\}$ of points in G-metric space X is said to be G-convergent to x if $\lim_{m,n\to\infty} G(x, x_n, x_m) = 0$; i.e. for each $\in > 0$ there exists a positive integer N_1 such that $G(x, x_n, x_m) < \in$ for all $m, n \geq N_1$. We say x is the limit of the sequence and write$x_n \to x$ or $\lim_{n\to\infty} x_n = x$."

Theorem 1.3 [75] "The following are equivalent in a G-metric space:

(i) $\{x_n\}$ is G-convergent to x;

(ii) $G(x_n, x_n, x) \to 0$ as $n \to \infty$;

(iii) $G(x_n, x, x) \to 0$ as $n \to \infty$;

(iv) $G(x_m, x_n, x) \to 0$ as $m, n \to \infty$."

Definition 1.4 [75]"Let (X, G) be a G-metric space. A sequence $\{x_n\}$ is called G-Cauchy if, for each $\in > 0$ there exists a positive integer N_1 such that

$$G(x_n, x_m, x_l) < \in \text{ for all } n, m, l \geq N_1."$$

Theorem 1.5 [75]"The following are equivalent in a G-metric space :

(i) the sequence $\{x_n\}$ is G-Cauchy,

(ii) for each $\in > 0$ there exists an N such that $G(x_n, x_m, x_l) < \in$ for all $n, m, l \geq N_1$."

Theorem 1.6 [75]"The function G(x, y, z) is jointly continuous in all three of its variables in a G-metric space."

Definition 1.7 [75]"A G-metric space (X, G) is called a symmetric G-metric space if G(x, y, y) = G(y, x, x) for all x, y in X."

Theorem 1.8 [75]"Every G-metric space (X, G) defines a metric space (X, d_G) by

$d_G(x, y) = G(x, y, y) + G(y, x, x)$ for all x, y in X.

If (X, G) is a symmetric G-metric space, then

$d_G(x, y) = 2G(x, y, y)$ for all x, y in X.

However, if (X, G) is not symmetric, then it follows from the G-metric properties that

$^3/_2 G(x, y, y) \leq d_G(x, y) \leq 3G(x, y, y)$ for all x, y in X."

Theorem 1.9 [75]"A G-metric space (X, G) is G-complete if and only if (X, d_G) is a complete metric space."

Theorem 1.10 [75]"Let (X, G) be a G-metric space. Then, for any x, y, z, a in X, it follows that:

(i) if $G(x, y, z) = 0$, then $x = y = z$;

(ii) $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$;

(iii) $G(x, y, y) \leq 2G(y, x, x)$;

(iv) $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$;

(v) $G(x, y, z) \leq {}^{2}/_{3}\, (G(x, a, a) + G(y, a, a) + G(z, a, a))$."

Definition 1.11 [75]"Let f and g be single-valued self-mappings on a set X. If $w = fx = gx$ for some $x \in X$, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g."

Definition 1.12 [51] "A pair (f, g) of self-mappings of a metric space (X, d) is said to be compatible if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some $z \in X$."

The weakly contractive mappings on Hilbert spaces was defined by Alber and Guerre-Delabriere as follows:

Definition 1.13[9] "A mapping $f : X \to X$ is said to be a weakly contractive mapping if $d(fx, fy) \leq d(x, y) - \varphi(d(x, y))$ for each $x, y \in X$ and $\varphi : [0,\infty) \to [0,\infty)$ is a continuous and non-decreasing function such that $\varphi(t) = 0$ if and only if $t = 0$."

Theorem 1.14 [90] "Let (X,d)be a complete metric space and $f : X \to X$ be a weakly contractive mapping. Then f has a unique fixed point."

Zhang and Song defined generalized φ – weak contractive condition as:

Definition 1.15 [113] "Two mappings $T, S : X \to X$ are called generalized φ-weak contractive if there exists a lower semi-continuous function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(t) = 0$ for $t = 0$ and $\varphi(t) > 0$ for all $t > 0$ such that

$$d(Tx, Sy) \leq N(x, y) - \varphi(N(x, y)) \text{ for each } x, y \in X,$$

where $N(x, y) = \max\{ d(x, y), d(x, Tx), d(y, Sy), \frac{1}{2}(d(x, Sy) + d(y, Tx))\}$."

Theorem 1.16 [113] "Let (X,d)be a complete metric space and $T, S : X \to X$ be generalized φ-weak contractive mappings, where $\varphi : [0,\infty) \to [0,\infty)$ is a lower semi-continuous function with

$\varphi(t) = 0$ for $t = 0$ and $\varphi(t) > 0$ for all $t > 0$. Then thereexists a unique fixed point $u \in X$ such that $u = Tu = Su$."

Definition 1.17[51]"Let (X,d) be a metric space and f, g: X→X be two mappings. The pair (f, g) is said to be compatible if and only if

$\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$,whenever$\{x_n\}$ is a sequence in X such that

$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$for some $t \in X$."

Definition 1.18 [61]"Let(X, G)be a G-metric space and f, g : X→X be two mappings. The pair (f, g) is said to be compatible if and only if $\lim_{n\to\infty} G(fgx_n, fgx_n, gfx_n)=0$, whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t, \text{ for some } t \in X."$$

Definition 1.19[52]"Let f and g be two self-mappings of a metric space(X, d). Then f and g are said to be weakly compatible if for all $x \in X$, the equality $fx=gx$ implies $fgx = gfx$."

2. Main Result

Theorem 2.1 Let f, g, h, A, B and C be six self-mappings in a complete G-metric space (X, G) satisfying the following conditions:

(i) $f(X) \subset B(X)$, $g(X) \subset C(X)$, $h(X) \subset A(X)$;

(ii) for all $x, y, z \in X$;

$$\int_0^{G(fx,gy,hz)} \phi(t)dt$$

$$\leq k\left(\max\left\{\begin{matrix}\int_0^{G(Ax,gy,gy)}\phi(t)dt+\int_0^{G(By,fx,fx)}\phi(t)dt+\int_0^{G(Ax,fx,fx)}\phi(t)dt,\\ \int_0^{G(By,hz,hz)}\phi(t)dt+\int_0^{G(Cz,gy,gy)}\phi(t)dt+\int_0^{G(By,gy,gy)}\phi(t)dt,\\ \int_0^{G(Cz,fx,fx)}\phi(t)dt+\int_0^{G(Ax,hz,hz)}\phi(t)dt+\int_0^{G(Cz,hz,hz)}\phi(t\,dt\end{matrix}\right\}\right) \quad (2.1)$$

or

$$\int_0^{G(fx,gy,hz)}\phi(t)dt\leq k\left(\max\left\{\begin{matrix}\int_0^{G(Ax,Ax,gy)}\phi(t)dt+\int_0^{G(By,By,fx)}\phi(t)dt+\int_0^{G(Ax,Ax,fx)}\phi(t)dt,\\ \int_0^{G(By,By,hz)}\phi(t)dt+\int_0^{G(Cz,Cz,gy)}\phi(t)dt+\int_0^{G(By,By,gy)}\phi(t)dt,\\ \int_0^{G(Cz,Cz,fx)}\phi(t)dt+\int_0^{G(Ax,Ax,hz)}\phi(t)dt+\int_0^{G(Cz,Cz,hz)}\phi(t)dt\end{matrix}\right\}\right) \quad (2.2)$$

where $k \in [0, \frac{1}{6})$. Then (f, A) or (g, B) or (h, C) has a coincidence point in X.

Moreover, if one of the following conditions is satisfied:

(a) f or A is G-continuous, (f, A) is weakly commuting, (g, B) and (h, C) are weakly compatible;

(b) g or B is G-continuous, (g, B) is weakly commuting, (f, A) and (h, C) are weakly compatible;

(c) h or C is G-continuous, (h, C) is weakly commuting, (f, A) and (g, B) are weakly compatible.

Then the mappings f, g, h, A, B and C have a unique common fixed point in X.

Proof. Let us first assume that mappings f, g, h, A, B and C satisfy condition (2.1). Let x_0 in X be an arbitrary point, since f (X) is contained in B(X), g(X) is contained in C(X) and h(X) is contained in A(X) there exist the sequences $\{x_n\}$ and $\{y_n\}$ in X, such that $y_{3n} = fx_{3n} = Bx_{3n+1}$, $y_{3n+1} = gx_{3n+1} = Cx_{3n+2}$, $y_{3n+2} = hx_{3n+2} = Ax_{3n+3}$ for n = 0, 1, 2, ….

If $y_{3n} = y_{3n+1}$, for some n, say n = 3m, then $p = x_{3m+1}$ is the coincidence point of (g, B). If $y_{3n+1} = y_{3n+2}$, for some n, say n = 3m, then $p = x_{3m+2}$ is the coincidence point of the pair (h, C). If $y_{3n+2} = y_{3n+3}$, for some n, say n = 3m, then $p = x_{3m+3}$ is the coincidence point of (f, A). So, we can assume that $y_n \neq y_{n+1}$, for all n = 0, 1, 2, ...

Now we prove that $\{y_n\}$ is a G-Cauchy sequence in X.

Since $G(y_{3n-1}, y_{3n}, y_{3n+1}) = G(y_{3n}, y_{3n+1}, y_{3n-1})$, using the condition (2.1) and (G-3) we have

$$\int_0^{G(y_{3n-1}, y_{3n}; y_{3n+1})} \phi(t)dt = \int_0^{G(fx_{3n}, gx_{3n+1}, hx_{3n-1})} \phi(t)dt$$

$$\leq k\left(\max\left\{\begin{array}{c} \int_0^{G(Ax_{3n}, gx_{3n+1}, gx_{3n+1})} \phi(t)dt + \int_0^{G(Bx_{3n+1}, fx_{3n}, fx_{3n})} \phi(t)dt \\ + \int_0^{G(Ax_{3n}, fx_{3n}, fx_{3n})} \phi(t)dt, \\ \int_0^{G(Bx_{3n+1}, hx_{3n-1}, hx_{3n-1})} \phi(t)dt + \int_0^{G(Cx_{3n-1}, gx_{3n+1}, gx_{3n+1})} \phi(t)dt \\ + \int_0^{G(Bx_{3n+1}, gx_{3n+1}, gx_{3n+1})} \phi(t)dt, \\ \int_0^{G(Cx_{3n-1}, fx_{3n}, fx_{3n})} \phi(t)dt + \int_0^{G(Ax_{3n}, hx_{3n-1}, hx_{3n-1})} \phi(t)dt \\ + \int_0^{G(Cx_{3n-1}, hx_{3n-1}, hx_{3n-1})} \phi(t)dt \end{array}\right\}\right)$$

$$= k\left(\max\left\{\begin{array}{c}\int_0^{G(y_{3n-1},y_{3n+1},y_{3n+1})}\phi(t)dt+\int_0^{G(y_{3n},y_{3n},y_{3n})}\phi(t)dt\\+\int_0^{G(y_{3n-1},y_{3n},y_{3n})}\phi(t)dt,\\\int_0^{G(y_{3n},y_{3n-1},y_{3n-1})}\phi(t)dt+\int_0^{G(y_{3n-2},y_{3n+1},y_{3n+1})}\phi(t)dt\\+\int_0^{G(y_{3n},y_{3n+1},y_{3n+1})}\phi(t)dt,\\\int_0^{G(y_{3n-2},y_{3n},y_{3n})}\phi(t)dt+\int_0^{G(y_{3n-1},y_{3n-1},y_{3n-1})}\phi(t)dt\\+\int_0^{G(y_{3n-2},y_{3n-1},y_{3n-1})}\phi(t)dt\end{array}\right\}\right)$$

$$\leq k\left(\max\left\{\begin{array}{c}\int_0^{G(y_{3n-1},y_{3n},y_{3n+1})}\phi(t)dt+0+\int_0^{G(y_{3n-2},y_{3n-1},y_{3n})}\phi(t)dt,\\\int_0^{G(y_{3n-1},y_{3n},y_{3n+1})}\phi(t)dt+\int_0^{G(y_{3n-2},y_{3n+1},y_{3n+1})}\phi(t)dt\\+\int_0^{G(y_{3n-1},y_{3n},y_{3n+1})}\phi(t)dt,\\\int_0^{G(y_{3n-2},y_{3n-1},y_{3n})}\phi(t)dt+0+\int_0^{G(y_{3n-2},y_{3n-1},y_{3n})}\phi(t)dt\end{array}\right\}\right)$$

$$\leq k\left(\max\left\{\begin{array}{c}\int_0^{G(y_{3n-1},y_{3n},y_{3n+1})}\phi(t)dt+\int_0^{G(y_{3n-2},y_{3n-1},y_{3n})}\phi(t)dt,\\ 3\int_0^{G(y_{3n-1},y_{3n},y_{3n+1})}\phi(t)dt+\int_0^{G(y_{3n-2},y_{3n-1},y_{3n})}\phi(t)dt,\\ 2\int_0^{G(y_{3n-2},y_{3n-1},y_{3n})}\phi(t)dt\end{array}\right\}\right)$$

$$\leq k\max\left\{3\int_0^{G(y_{3n-1},y_{3n},y_{3n+1})}\phi(t)dt+3\int_0^{G(y_{3n-2},y_{3n-1},y_{3n})}\phi(t)dt\right\}$$

$$=3k\left(\int_0^{G(y_{3n-1},y_{3n},y_{3n+1})}\phi(t)dt+\int_0^{G(y_{3n-2},y_{3n-1},y_{3n})}\phi(t)dt\right),$$

which further implies that

$$(1-3k)\int_0^{G(y_{3n-1},y_{3n},y_{3n+1})}\phi(t)dt\leq 3k\int_0^{G(y_{3n-2},y_{3n-1},y_{3n})}\phi(t)dt.$$

$$=\alpha\int_0^{G(y_{3n-2},y_{3n-1},y_{3n})}\phi(t)dt,$$

where $\alpha=\frac{3k}{1-3k}$. Obviously $0\leq\alpha<1$.

Similarly, it can be shown that

$$\int_0^{G(y_{3n},y_{3n+1},y_{3n+2})}\phi(t)dt\leq\alpha\int_0^{G(y_{3n-1},y_{3n},y_{3n+1})}\phi(t)dt$$

and

$$\int_0^{G(y_{3n+1},y_{3n+2},y_{3n+3})}\phi(t)dt\leq\alpha\int_0^{G(y_{3n},y_{3n+1},y_{3n+2})}\phi(t)dt.$$

It follows that for all $n\in\mathbb{N}$,

$$\int_0^{G(y_n,y_{n+1},y_{n+2})}\phi(t)dt\leq\alpha\int_0^{G(y_{n-1},y_n,y_{n+1})}\phi(t)dt\leq\alpha^2\int_0^{G(y_{n-2},y_{n-1},y_n)}\phi(t)dt$$

$$\leq\ldots\leq\alpha^n\int_0^{G(y_0,y_1,y_2)}\phi(t)dt.$$

Therefore, for all $n, m\in\mathbb{N}$, $n<m$, we have

$$\int_0^{G(y_n,y_m,y_m)}\phi(t)dt\leq\int_0^{G(y_n,y_{n+1},y_{n+1})}\phi(t)dt+\int_0^{G(y_{n+1},y_{n+2},y_{n+2})}\phi(t)dt$$

$$+\int_0^{G(y_{n+2},y_{n+3},y_{n+3})}\phi(t)dt+\ldots+\int_0^{G(y_{m-1},y_m,y_m)}\phi(t)dt$$

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$\leq \int_0^{G(y_n, y_{n+1}, y_{n+2})} \phi(t)dt + \int_0^{G(y_{n+1}, y_{n+2}, y_{n+3})} \phi(t)dt$

$+ \ \ldots \ + \int_0^{G(y_{m-1}, y_m, y_{m+1})} \phi(t)dt$

$$\leq (\alpha^n + \alpha^{n+1} + \alpha^{n+2} + \quad + \alpha^{m-1}) \int_0^{G(y_0, y_1, y_2)} \phi(t)dt$$

$$\leq \frac{\alpha^n}{1-\alpha} \int_0^{G(y_0, y_1, y_2)} \phi(t)dt \to 0 \text{ as } n \to \infty.$$

Hence, $\{y_n\}$ is a G-Cauchy sequence in a complete G-metric space X.

So there exists a point $u \in X$ such that $y_n \to u$ as $n \to \infty$.The sub sequences of $\{y_n\}$ viz $\{fx_{3n}\}=\{Bx_{3n+1}\}$, $\{gx_{3n+1}\}=\{Cx_{3n+2}\}$ and $\{hx_{3n-1}\}=\{Ax_{3n}\}$ are all convergent to u, that is

$y_{3n} = fx_{3n} = Bx_{3n+1} \to u, y_{3n+1} = gx_{3n+1} = Cx_{3n+2} \to u;$

$$y_{3n-1} = hx_{3n-1} = Ax_{3n} \to u \text{ as } n \to \infty. \quad (2.3)$$

Now we prove that mappings f, g, h, A, B and C under the condition (a) of our theorem have u as a common fixed point.

Firstly, let A is continuous, (f, A) is weakly commuting, (g, B) and (h, C) are weakly compatible.

Step 1. We prove that u = fu = Au.

By (2.3) and using weakly commuting of (f, A), we get

$$G(fAx_{3n}, Afx_{3n}, Afx_{3n}) \leq G(fx_{3n}, Ax_{3n}, Ax_{3n}) \to 0 \text{ as } n \to \infty. \quad (2.4)$$

Since A is continuous, then $A^2x_{3n} \to Au$ as $n \to \infty$, $Afx_{3n} \to Au$ as $n \to \infty$.

By (2.4), we know $fAx_{3n} \to$ Auas $n \to \infty$.

From the condition (2.1) we know:

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$\int_0^{G(fAx_{3n},gx_{3n+1},hx_{3n+2})} \phi(t)dt \leq$

$$k\left(\max\left\{\begin{array}{c}\int_0^{G(AAx_{3n},gx_{3n+1},gx_{3n+1})}\phi(t)dt+\int_0^{G(Bx_{3n+1},fAx_{3n},fAx_{3n})}\phi(t)dt\\ +\int_0^{G(AAx_{3n},fAx_{3n},fAx_{3n})}\phi(t)dt,\\ \int_0^{G(Bx_{3n+1},hx_{3n+2},hx_{3n+2})}\phi(t)dt+\int_0^{G(Cx_{3n+2},gx_{3n+1},gx_{3n+1})}\phi(t)dt\\ +\int_0^{G(Bx_{3n+1},gx_{3n+1},gx_{3n+1})}\phi(t)dt,\\ \int_0^{G(Cx_{3n+2},fAx_{3n},fAx_{3n})}\phi(t)dt+\int_0^{G(AAx_{3n},hx_{3n+2},hx_{3n+2})}\phi(t)dt\\ +\int_0^{G(Cx_{3n+2},hx_{3n+2},hx_{3n+2})}\phi(t)dt\end{array}\right\}\right).$$

Letting $n \to \infty$, and using the Theorem 1.10 (iii), we have

$$\int_0^{G(Au,u,u)} \phi(t)dt$$

$$\leq k\left(\max\left\{\begin{array}{c}\int_0^{G(Au,u,u)}\phi(t)dt+\int_0^{G(u,Au,Au)}\phi(t)dt+\int_0^{G(Au,Au,Au)}\phi(t)dt,\\ \int_0^{G(uu,u,u)}\phi(t)dt+\int_0^{G(u,u,u)}\phi(t)dt+\int_0^{G(u,u,u)}\phi(t)dt,\\ \int_0^{G(u,Au,Au)}\phi(t)dt+\int_0^{G(Au,u,u)}\phi(t)dt+\int_0^{G(u,u,u)}\phi(t)dt\end{array}\right\}\right)$$

$$= k\left(\int_0^{G(Au,u,u)}\phi(t)dt + \int_0^{G(u,Au,Au)}\phi(t)dt,\right) \leq 3k\int_0^{G(Au,u,u)}\phi(t)dt.$$

Hence, $G(Au, u, u)=0$ and $Au = u$, since $k \in [0,\frac{1}{6})$ and for each $\epsilon > 0$,

$\int_0^{\epsilon} \phi(t)dt > 0.$

Use of the condition (2.1) gives

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$$\int_0^{G(fu,gx_{3n+1},hx_{3n+2})} \phi(t)dt$$

$$\leq k\left(\max\left\{\begin{array}{c}\int_0^{G(Au,gx_{3n+1},gx_{3n+1})}\phi(t)dt+\int_0^{G(Bx_{3n+1},fu,fu)}\phi(t)dt\\ +\int_0^{G(Au,fu,fu)}\phi(t)dt,\\ \int_0^{G(Bx_{3n+1},hx_{3n+2},hx_{3n+2})}\phi(t)dt+\int_0^{G(Cx_{3n+2},gx_{3n+1},gx_{3n+1})}\phi(t)dt\\ +\int_0^{G(Bx_{3n+2},gx_{3n+1},gx_{3n+1})}\phi(t)dt,\\ \int_0^{G(Cx_{3n+2},fu,fu)}\phi(t)dt+\int_0^{G(Au,hx_{3n+2},hx_{3n+2})}\phi(t)dt\\ +\int_0^{G(Cx_{3n+2},hx_{3n+2},hx_{3n+2})}\phi(t)dt\end{array}\right\}\right).$$

Letting $n \to \infty$, using Au = u we have

$$\int_0^{G(fu,u,u)}\phi(t)dt \leq k\int_0^{G(u,fu,fu)}\phi(t)dt.$$

Using Theorem 1.10(iii), we get

$$\int_0^{G(fu,u,u)}\phi(t)dt \leq 2k\int_0^{G(fu,u,u)}\phi(t)dt$$

which gives G(fu, u, u)=0 and so fu = u, since $k \in [0, \frac{1}{6})$ and for each $\epsilon > 0$,

$\int_0^{\epsilon}\phi(t)dt>0$ .Thus we have u = Au = fu.

Step 2. We now show u = gu = Bu.

As f (X) is contained in B(X) and u = fu ∈ f (X), there exists is a point v ∈ X such that u = fu = Bv.

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Use of condition (2.1) yields

$$\int_0^{G(fu,gv,hx_{3n+2})} \phi(t)dt$$

$$\leq k\left(\max\left\{\begin{array}{c} \int_0^{G(Au,gv,gv)} \phi(t)dt + \int_0^{G(Bv,fu,fu)} \phi(t)dt \\ + \int_0^{G(Au,fu,fu)} \phi(t)dt, \\ \int_0^{G(Bv,hx_{3n+2},hx_{3n+2},)} \phi(t)dt + \int_0^{G(Cx_{3n+2},,gv,gv)} \phi(t)dt \\ + \int_0^{G(Bv,gv,gv)} \phi(t)dt, \\ \int_0^{G(Cx_{3n+2},fu,fu)} \phi(t)dt + \int_0^{G(Au,hx_{3n+2},hx_{3n+2})} \phi(t)dt \\ + \int_0^{G(Cx_{3n+2},hx_{3n+2},hx_{3n+2})} \phi(t)dt \end{array}\right\}\right).$$

Letting $n \to \infty$, using $u = Au = fu = Bv$ and by Theorem 1.10(iii), we obtain

$$\int_0^{G(u,gv,u)} \phi(t)dt \leq 2\int_0^{G(u,gv,gv)} \phi(t)dt$$

$$\leq 3k\int_0^{G(u,gv,u)} \phi(t)dt,$$

which gives that $G(u, gv, u) = 0$ because $k \in [0,\frac{1}{6})$ and for each $\epsilon > 0$, $\int_0^{\epsilon} \phi(t)dt > 0$, and so $gv = u = Bv$. Since (g, B) is weakly compatible, we get

$gu = gBv = Bgv = Bu$.

Again by (2.1),

$$\int_0^{G(fu,gu,hx_{3n+2})} \phi(t)dt$$

$$\leq k\left(\max\left\{\begin{array}{c}\int_0^{G(Au,gu,gu)}\phi(t)dt+\int_0^{G(Bu,fu,fu)}\phi(t)dt\\+\int_0^{G(Au,fu,fu)}\phi(t)dt,\\\int_0^{G(Bu,hx_{3n+2},hx_{3n+2})}\phi(t)dt+\int_0^{G(Cx_{3n+2},gu,gu)}\phi(t)dt\\+\int_0^{G(Bu,gu,gu)}\phi(t)dt,\\\int_0^{G(Cx_{3n+2},fu,fu)}\phi(t)dt+\int_0^{G(Au,hx_{3n+2},hx_{3n+2})}\phi(t)dt\\+\int_0^{G(Cx_{3n+2},hx_{3n+2},hx_{3n+2})}\phi(t)dt\end{array}\right\}\right).$$

Letting $n \to \infty$, using $u = Au = fu$, $gu = Bu$ and by Theorem 1.10 (iii),

$$\int_0^{G(u,gu,u)}\phi(t)dt \leq k\left(\max\left\{\begin{array}{c}\int_0^{G(u,gu,gu)}\phi(t)dt+\int_0^{G(gu,u,u)}\phi(t)dt+\int_0^{G(u,u,u)}\phi(t)dt,\\\int_0^{G(gu,u,u)}\phi(t)dt+\int_0^{G(u,gu,gu)}\phi(t)dt+\int_0^{G(gu,gu,gu)}\phi(t)dt,\\\int_0^{G(u,u,u)}\phi(t)dt+\int_0^{G(u,u,u)}\phi(t)dt+\int_0^{G(u,u,u)}\phi(t)dt\end{array}\right\}\right)$$

$$\leq k\left(\max\left\{\int_0^{G(u,gu,gu)}\phi(t)dt+\int_0^{G(gu,u,u)}\phi(t)dt\right\}\right)$$

$$\leq 3k\int_0^{G(u,gu,u)}\phi(t)dt,$$

which gives G(u, gu, u) = 0 since for each $\epsilon > 0$, $\int_0^{\epsilon} \phi(t)dt > 0$ and so u = gu = Bu.

Step 3. We show that u = hu = Cu.

As g(X) is contained in C(X) and u = gu∈ g(X), there exists a point w ∈X such that u = gu = Cw. Again by use of condition (2.1), we have

$$\int_0^{G(fu,gu,hw)} \phi(t)dt \leq k\left(\max\left\{\begin{array}{l} \int_0^{G(Au,gu,gu)} \phi(t)dt + \int_0^{G(Bu,fu,fu)} \phi(t)dt + \int_0^{G(Au,fu,fu)} \phi(t)dt, \\ \int_0^{G(Bu,hw,hw)} \phi(t)dt + \int_0^{G(Cw,gu,gu)} \phi(t)dt + \int_0^{G(Bu,gu,gu)} \phi(t)dt, \\ \int_0^{G(Cw,fu,fu)} \phi(t)dt + \int_0^{G(Au,hw,hw)} \phi(t)dt + \int_0^{G(Cw,hw,hw)} \phi(t)dt \end{array}\right\}\right).$$

Using u = Au = fu, u = gu = Bu = Cw and Theorem 1.10 (iii), we obtain

$$\int_0^{G(fu,gu,hw)} \phi(t)dt = \int_0^{G(u,u,hw)} \phi(t)dt$$

$$\leq k\int_0^{G(u,hw,hw)} \phi(t)dt \leq 2k\int_0^{G(u,u,hw)} \phi(t)dt$$

giving that G(u, u, hw) = 0 and so hw = u = Cw.

Weak compatibility of (h, C) implies hu = hCw = Chw = Cu.

Using (2.1), we get

$$\int_0^{G(fu,gu,hu)} \phi(t)dt$$

$$\leq k\left(\max\left\{\begin{array}{l}\int_0^{G(Au,gu,gu)}\phi(t)dt+\int_0^{G(Bu,fu,fu)}\phi(t)dt+\int_0^{G(Au,fu,fu)}\phi(t)dt,\\ \int_0^{G(Bu,hu,hu)}\phi(t)dt+\int_0^{G(Cu,hu,hu)}\phi(t)dt+\int_0^{G(Bu,gu,gu)}\phi(t)dt,\\ \int_0^{G(Cu,fu,fu)}\phi(t)dt+\int_0^{G(Au,hu,hu)}\phi(t)dt+\int_0^{G(Cu,hu,hu)}\phi(t)dt\end{array}\right\}\right).$$

As $u = Au = fu$, $u = gu = Bu$, $Cu = hu$ and the Theorem 1.10 (iii), we get

$$\int_0^{G(u,u,hu)}\phi(t)dt \leq k\max\left\{\int_0^{G(u,hu,hu)}\phi(t)dt+\int_0^{G(hu,u,u)}\phi(t)dt\right\}$$

$$\leq 3k\int_0^{G(u,u,hu)}\phi(t)dt,$$

which gives that $G(u, u, hu) = 0$ since for each $\epsilon > 0$, $\int_0^{\epsilon}\phi(t)dt > 0$ and so $u = hu = Cu$. Therefore, if A is continuous, (f, A) is weakly commuting, (g, B) and (h, C) are weakly compatible, then f , g, h, A, B and C have a common fixed point u.

Next let f be continuous, (f, A) is weakly commuting, (g, B) and (h, C) are weakly compatible.

Step 4. We show that u = fu.

Using (2.3) and weakly commuting of (f, A), we have

$$\int_0^{G(fAx_{3n},Afx_{3n},Afx_{3n})}\phi(t)dt \leq \int_0^{G(fx_{3n},Ax_{3n},Ax_{3n})}\phi(t)dt \to 0 \text{ as } n \to \infty.$$

Since f is continuous, then $f^2x_{3n} \to fu$ as $n \to \infty$, $fAx_{3n} \to fu$ as $n \to \infty$.

Using (2.3) we know $Afx_{3n} \to fu$ as $n \to \infty$.

From the condition (2.1) we have

$$\int_0^{G(f^2x_{3n},gx_{3n+1},hx_{3n+2})}\phi(t)dt$$

$$\leq k\left(\max\left\{\begin{array}{c}\int_0^{G(Afx_{3n},gx_{3n+1},gx_{3n+1})}\phi(t)dt+\int_0^{G(Bx_{3n+1},ffx_{3n},ffx_{3n})}\phi(t)dt\\+\int_0^{G(Afx_{3n},ffx_{3n},ffx_{3n})}\phi(t)dt,\\\int_0^{G(Bx_{3n},hx_{3n+2},hx_{3n+2})}\phi(t)dt+\int_0^{G(Cx_{3n+2},gx_{3n+1},gx_{3n+1})}\phi(t)dt\\+\int_0^{G(Bx_{3n+1},gx_{3n+1},gx_{3n+1})}\phi(t)dt,\\\int_0^{G(Cx_{3n+2},ffx_{3n+2},ffx_{3n})}\phi(t)dt+\int_0^{G(Afx_{3n},hx_{3n+2},hx_{3n+2})}\phi(t)dt\\+\int_0^{G(Cx_{3n+2},hx_{3n+2},hx_{3n+2})}\phi(t)dt\end{array}\right\}\right).$$

Letting $n\to\infty$ and by the Theorem 1.10 (iii), we have

$$\int_0^{G(fu,u,u)}\phi(t)dt$$

$$\leq k\left(\max\left\{\begin{array}{c}\int_0^{G(fu,u,u)}\phi(t)dt+\int_0^{G(u,fu,fu)}\phi(t)dt+\int_0^{G(fu,fu,fu)}\phi(t)dt,\\\int_0^{G(uu,u,u)}\phi(t)dt+\int_0^{G(u,u,u)}\phi(t)dt+\int_0^{G(u,u,u)}\phi(t)dt,\\\int_0^{G(u,fu,fu)}\phi(t)dt+\int_0^{G(u,u,u)}\phi(t)dt+\int_0^{G(u,u,u)}\phi(t)dt\end{array}\right\}\right)$$

$$= k\left(\max\left\{\int_0^{G(fu,u,u)}\phi(t)dt+\int_0^{G(u,fu,fu)}\phi(t)dt\right\}\right)$$

$$\leq 3k\int_0^{G(fu,u,u)}\phi(t)dt,$$

resulting that G(fu, u, u) = 0, since for each $\epsilon > 0$, $\int_0^{\epsilon} \phi(t)dt > 0$ and so fu = u.

Step 5. We prove that u = gu = Bu.

Since f (X) is contained in B(X) and u = fu ∈ f (X), there exists a point z ∈ X such that u = fu = Bz. Use of condition (2.1) yields

$$\int_0^{G(f^2x_{3n},gz,hx_{3n+2})} \phi(t)dt$$

$$\leq k\left(\max\left\{\begin{array}{c}\int_0^{G(Afx_{3n},gz,gz)}\phi(t)dt+\int_0^{G(Bz,ffx_{3n},ffx_{3n})}\phi(t)dt\\ +\int_0^{G(Afx_{3n},ffx_{3n},ffx_{3n})}\phi(t)dt,\\ \int_0^{G(Bz,hx_{3n+2},hx_{3n+2})}\phi(t)dt+\int_0^{G(Cx_{3n+2},gz,gz)}\phi(t)dt\\ +\int_0^{G(Bz,gz,gz)}\phi(t)dt,\\ \int_0^{G(Cx_{3n+2},ffx_{3n},ffx_{3n})}\phi(t)dt+\int_0^{G(Ax_{3n},hx_{3n+2},hx_{3n+2})}\phi(t)dt\\ +\int_0^{G(Cx_{3n+2},hx_{3n+2},hx_{3n+2})}\phi(t)dt\end{array}\right\}\right).$$

Letting n → ∞, using u = fu = Bz and the Theorem 1.10 (iii) we have

$$\int_0^{G(u,gz,u)} \phi(t)dt \leq 3k \int_0^{G(u,gz,u)} \phi(t)dt$$

which implies that G(u, gz, u) = 0 and so gz = u = Bz.

Since (g, B) is weakly compatible, so gu = gBz = Bgz = Bu.

Use of condition (2.1) gives

$$\int_0^{G(fx_{3n},gu,hx_{3n+2})} \phi(t)dt$$

$$\leq k\left(\max\left\{\begin{array}{c}\int_0^{G(Ax_{3n},gu,gu)}\phi(t)dt+\int_0^{G(Bu,fx_{3n},fx_{3n})}\phi(t)dt \\ +\int_0^{G(Ax_{3n},fx_{3n},fx_{3n})}\phi(t)dt, \\ \int_0^{G(Bu,hx_{3n+2},hx_{3n+2})}\phi(t)dt+\int_0^{G(Cx_{3n+2},gu,gu)}\phi(t)dt \\ +\int_0^{G(Bu,gu,gu)}\phi(t)dt, \\ \int_0^{G(Cx_{3n+2},fx_{3n},fx_{3n})}\phi(t)dt+\int_0^{G(Ax_{3n},hx_{3n+2},hx_{3n+2})}\phi(t)dt \\ +\int_0^{G(Cx_{3n+2},hx_{3n+2},hx_{3n+2})}\phi(t)dt\end{array}\right\}\right).$$

Letting $n \to \infty$, using $u = fu$, $gu = Bu$ and Theorem 1.10 (iii) we have

$$\int_0^{G(u,gu,u)}\phi(t)dt \leq k\left\{\int_0^{G(u,gu,gu)}\phi(t)dt+\int_0^{G(gu,u,u)}\phi(t)dt\right\}$$

$$\leq 3k\int_0^{G(u,gu,u)}\phi(t)dt,$$

which is not possible and so $G(u, gu, u) = 0$, since for each $\epsilon > 0$, $\int_0^{\epsilon}\phi(t)dt > 0$ and

so $gu = u = Bu$.

Step 6. We show that $u = hu = Cu$.

Since $g(X)$ is contained in $C(X)$ and $u = gu \in g(X)$, there exists a point $t \in X$ such that $u = gu = Ct$.

Use of condition (2.1) gives

$$\int_0^{G(fx_{3n},gu,ht)}\phi(t)dt$$

$$\leq k\left(\max\left\{\begin{array}{c}\int_0^{G(Ax_{3n},gu,gu)}\phi(t)dt+\int_0^{G(Bu,fx_{3n},fx_{3n})}\phi(t)dt\\+\int_0^{G(Ax_{3n},fx_{3n},fx_{3n})}\phi(t)dt,\\ \int_0^{G(Bu,ht,ht)}\phi(t)dt+\int_0^{G(Ct,gu,gu)}\phi(t)dt\\+\int_0^{G(Bu,gu,gu)}\phi(t)dt,\\ \int_0^{G(Ct,fx_{3n},fx_{3n})}\phi(t)dt+\int_0^{G(Ax_{3n},ht,ht)}\phi(t)dt\\+\int_0^{G(Ct,ht,ht)}\phi(t)dt\end{array}\right\}\right).$$

Letting $n \to \infty$, using $u = gu = Bu = Ct$ and by Theorem 1.10 (iii), we obtain

$\int_0^{G(u,u,ht)}\phi(t)dt \leq 3k\int_0^{G(u,u,ht)}\phi(t)dt$.

Hence $G(u, u, ht) = 0$ and so $ht = u = Ct$.

Since (h,C) is weakly compatible, we have $hu = hCt = Cht = Cu$.

Using (2.1), we have

$$\int_0^{G(fx_{3n},gu,hu)}\phi(t)dt$$

$$\leq k\left(\max\left\{\begin{array}{c}\int_0^{G(Ax_{3n},gu,gu)}\phi(t)dt+\int_0^{G(Bu,fx_{3n},fx_{3n})}\phi(t)dt\\+\int_0^{G(Ax_{3n},fx_{3n},fx_{3n})}\phi(t)dt,\\ \int_0^{G(Bu,hu,hu)}\phi(t)dt+\int_0^{G(u,gu,gu)}\phi(t)dt\\+\int_0^{G(Bu,gu,gu)}\phi(t)dt,\\ \int_0^{G(Cu,fx_{3n},fx_{3n})}\phi(t)dt+\int_0^{G(Ax_{3n},hu,hu)}\phi(t)dt\\+\int_0^{G(Cu,hu,hu)}\phi(t)dt\end{array}\right\}\right).$$

Letting $n\to\infty$, using u = gu = Bu, Cu = hu and by Theorem 1.10 (iii),

$$\int_0^{G(u,u,hu)}\phi(t)dt\leq k\left(\max\left\{\int_0^{G(u,hu,hu)}\phi(t)dt+\int_0^{G(hu,u,u)}\phi(t)dt\right\}\right)$$

$$\leq 3k\int_0^{G(u,u,hu)}\phi(t)dt,$$

which gives that G(u,u, hu) = 0, since for each $\epsilon > 0$, $\int_0^{\epsilon}\phi(t)dt > 0$ and

so hu = u = Cu.

Step 7. We show that u = Au. As h(X) is contained in A(X) and u = hu ∈ h(X), there exists a point p ∈ X with u = hu = Ap.

Using condition (2.1),

$$\int_0^{G(fp,gu,hu)}\phi(t)dt$$

$$\leq k\left(\max\left\{\begin{array}{c}\int_0^{G(Ap,gu,gu)}\phi(t)dt+\int_0^{G(Bu,fp,fp)}\phi(t)dt+\int_0^{G(Ap,fp,fp)}\phi(t)dt,\\ \int_0^{G(Bu,hu,hu)}\phi(t)dt+\int_0^{G(Cu,gu\ gu)}\phi(t)dt+\int_0^{G(Bu,gu,gu)}\phi(t)dt,\\ \int_0^{G(Cu,fp,fp)}\phi(t)dt+\int_0^{G(Ap,hu,hu)}\phi(t)dt+\int_0^{G(Cu,hu,hu)}\phi(t)dt\end{array}\right\}\right).$$

Taking u = gu = Bu, u = hu = Cu and the Theorem1.10 (iii), we get

$$\int_0^{G(fp,u,u)}\phi(t)dt\leq 3k\int_0^{G(fp,u,u)}\phi(t)dt\ ,$$

yielding G(fp, u, u) = 0, since for each $\epsilon > 0$, $\int_0^{\epsilon}\phi(t)dt > 0$ and so fp = u = Ap.

Since the pair (f, A) is weakly compatible, we havefu = fAp = Afp = Au = u.Therefore, if f is continuous, (f, A) is weakly commuting, (g, B) and (h, C) are weakly compatible then u is the common fixed point of f, g, h, A, B and C.

On the same lines, the result follows under the conditions (b) or (c) of our theorem.

Now we show uniqueness of common fixed point u.

Let u and q are two common fixed points of the mappings. Use of condition (2.1) gives

$$\int_0^{G(q,u,u)}\phi(t)dt = \int_0^{G(fq,gu,hu)}\phi(t)dt$$

$$\leq k\left(\max\left\{\begin{array}{l}\int_0^{G(Aq,gu,gu)}\phi(t)dt + \int_0^{G(Bu,fq,fq)}\phi(t)dt + \int_0^{G(Aq,fq,fq)}\phi(t)dt, \\ \int_0^{G(Bu,hu,hu)}\phi(t)dt + \int_0^{G(Cu,gu\ gu)}\phi(t)dt + \int_0^{G(Bu,gu,gu)}\phi(t)dt, \\ \int_0^{G(Cu,fq,fq)}\phi(t)dt + \int_0^{G(Aq,hu,hu)}\phi(t)dt + \int_0^{G(Cu,hu,hu)}\phi(t)dt\end{array}\right\}\right)$$

$$\leq k\left(\max\left\{\int_0^{G(u,q,q)}\phi(t)dt + \int_0^{G(q,u,u)}\phi(t)dt\right\}\right)$$

$$\leq 3k\int_0^{G(q,u,u)}\phi(t)dt,$$

giving G(q, u, u) = 0, since for each $\epsilon > 0$, $\int_0^{\epsilon}\phi(t)dt > 0$ and so q = u is a unique common fixed point . Condition (2.2) yields the same results.

If we take $\phi(t) = 1$ in Theorem 2.1, then it extends the following resultof Gu [40]

Corollary 2.2 [40] "Let (X, G) be a complete G-metric space and let f , g, h, A, B, and C be six mappings of X into itself satisfying the following conditions:

(i) $f(X) \subset B(X)$, $g(X) \subset C(X)$, $h(X) \subset A(X)$;

(ii) for all $x, y, z \in X$,

$$G(fx, gy, hz) \leq k\left(\max\begin{Bmatrix} G(Ax, gy, gy) + G(By, fx, fx), \\ G(By, hz, hz) + G(Cz, gy, gy), \\ G(Cz, fx, fx) + G(Ax, hz, hz) \end{Bmatrix}\right)$$

or

$$G(fx, gy, hz) \leq k\left(\max\begin{Bmatrix} G(Ax, Ax, gy) + G(By, By, fx), \\ G(By, By, hz) + G(Cz, Cz, gy), \\ G(Cz, Cz, fx) + G(Ax, Ax, hz) \end{Bmatrix}\right)$$

where $k \in [0, \frac{1}{3})$. Then one of the pairs (f, A), (g, B) and (h, C) has a coincidence point in X. Moreover, if one of the following conditions is satisfied:

(a) either f or A is G-continuous, the pair (f, A) is weakly commuting, the pairs (g, B) and (h, C) are weakly compatible;

(b) either g or B is G-continuous, the pair (g, B) is weakly commuting, the pairs (f, A) and (h, C) are weakly compatible;

(c) either h or C is G-continuous, the pair (h, C) is weakly commuting, the pairs (f, A) and (g, B) are weakly compatible.

Then the mappings f, g, h, A, B, and C have a unique common fixed point in X."

If we take ϕ (t) = 1 and A = B = C = I (identity map) in Theorem 2.1, it extends the following result of Abbas et. al [6] of three self-mappings.

Corollary 2.3 [6] "Let (X, G) be a complete G-metric space and let f, g and h be three mappings of X into itself satisfying the following condition:

$$G(fx, gy, hz) \leq k\left(\max\begin{Bmatrix} G(x, gy, gy) + G(y, fx, fx), \\ G(y, hz, hz) + G(z, gy, gy), \\ G(z, fx, fx) + G(x, hz, hz) \end{Bmatrix}\right)$$

for all x, y, z ∈ X, where $k \in [0, \frac{1}{3})$.Then the mappings f, g and h have a unique common fixed point in X."

If we take ϕ (t) = 1, A = B = C = I (identity map) and f = g = h in Theorem 2.1 it extends the result of Mustafa and Sims [76].

Corollary 2.4 [76] "Let (X, G) be a complete G-metric space and let f be a mapping of X into itself satisfying the following condition:

$$G(fx, fy, fz) \leq k\left(\max\left\{\begin{matrix} G(x, fy, fy) + G(y, fx, fx), \\ G(y, fz, fz) + G(z, fy, fy), \\ G(z, fx, fx) + G(x, fz, fz) \end{matrix}\right\}\right)$$

for all x, y, z ∈ X, where $k \in [0, \frac{1}{2})$.Then the mapping f has a unique fixed point in X."

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