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A NON STANDARD LOCAL APPROACH TO PROVING THE RICHARDSON EXTRAPOLATION IN THE FINITE ELEMENT METHOD BY PARTIAL DIFFERENTIAL EQUATION

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Abstract

In order to improve the precision of the traditional finite element approximation of solutions, this work provides a nonstandard local approach to Richardson extrapolation. We need a local condition that connects the several extrapolation subspaces in order to use this inequality to confirm that the Richardson method operates at a point. This condition, which may be roughly translated to mean that the subspaces are similar about a point; states that any one of them can be made to locally coincide with another by simply scaling the independent variable about that point. Examples of finite element subspaces that actually exist and meet this requirement are provided.

Keywords: Richardson Extrapolation, Finite Element Method

1. Introduction:

A singly perturbed delayed partial differential equation with modest spatial shift right boundary layer problem is decomposed into three piecewise equations in this study, and fitted operator difference methods are used to solve them. The Richardson extrapolation method is used to speed up the order of precision in the time variable. In both time and space, the present method's order of convergence is demonstrated to be second order, while the rate of convergence is two. Furthermore, the numerical findings of the examples analyzed reveal that the current method is more accurate than certain previously published results.

The Richardson extrapolation method has been defined, and it is intended to improve the accuracy of the basic scheme's computed solutions.

Let $D_2^n \subseteq D_2^{2n}$ where D_2^{2n} is the mesh created by cutting the step size k in half. The numerical solution found from is denoted byD_2^{2n} by $\bar{U}^j\,(x), we\ get$

$$u^j(x) - U^j(x) \leq Ck + R_j^n(x),\ (x, t_j) \in D_1 \times D_2^n, \qquad (1)$$

$$u^j(x) - \bar{U}^j(x) \leq C\left(\frac{k}{2}\right) + R_j^{2n}(x),\ (x, t_j) \in D_1 \times D_2^{2n}, \qquad (2)$$

Where $R_j^n\,(x) and\ R_j^{2n}\,(x)$are the error's remaining terms? To get the extrapolation formula, subtract the inequality (1) from (2).

$$u^j(x) - U^j(x) - 2\left(u^j(x) - \bar{U}^j(x)\right) = R_j^n(x) - R_j^{2n}, \qquad (3)$$

Which results in -

$$U_j^{\text{ext}}(x) = 2\bar{U}^j(x) - U^j(x), \qquad (4)$$

is a close approximation.

Theorem:$Let\ u(x_i, t_{j+1}) and\ U_{i,j+1}^{ext}$If (1) and (2) are solved, then the suggested scheme meets the following error estimate.

$$\sup_{0<\varepsilon<1} \max_{x_i, t_{j+1}} |u(x_i, t_{j+1}) - U_{i,j+1}^{\text{ext}}| \leq C(h^2 + k^2). \qquad (5)$$

Proof. The needed bound is obtained by using the error for both temporal and spatial discretization.

2. NUMERICAL ILLUSTRATION

We looked at model problems that had been addressed in the literature and had approximate answers that could be compared to determine the efficacy of the current scheme. When the exact answer for the given problem was uncertain, we used the double-mesh concept to estimate the absolute maximum inaccuracy of the current technique. To approximate the absolute maximum error at the given mesh locations, we apply the formula below:

Case: If you know the exact solution,

$$E_\varepsilon^{M,N} = \max_{(x_i,t_j)\in\Omega} \left| u(x_i,t_j) - u_i^{\text{ext},j} \right|. \tag{6}$$

Case: If the precise solution isn't known,

$$E_\varepsilon^{M,N} = \max_{(x_i,t_j)\in\Omega} \left| \left(u_i^{\text{ext},j}\right)^{M,N} - \left(u_i^{\text{ext},j}\right)^{2M,2N} \right|. \tag{7}$$

Table 1: Example 1's maximum absolute point-wise error and rate of convergence before and after using the Richardson extrapolation method, where γ = 0:5ε, μ = 0:6ε, and L ≥ 8.

ε↓M,N⟶ **Before**	32,32	64,64	128,128	256,256	512,512
10^{-2}	5.5909*e*-03	2.7792*e*-03	1.3847*e*-03	6.9081*e*-04	3.4501*e*-04
10^{-4}	6.2590*e*-03	3.3063*e*-03	1.5449*e*-03	7.7016*e*-04	3.8448*e*-04
10^{-6}	6.2656*e*-03	3.1096*e*-03	1.5465*e*-03	7.7096*e*-04	3.8488*e*-04
10^{-8}	6.2657*e*-03	3.1096*e*-03	1.5465*e*-03	7.7097*e*-04	3.8488*e*-04
10^{-10}	6.2657*e*-03	3.1096*e*-03	1.5465*e*-03	7.7097*e*-04	3.8488*e*-04
⋮	⋮	⋮	⋮	⋮	⋮
10^{-L}	6.2657*e*-03	3.1096*e*-03	1.5465*e*-03	7.7097*e*-04	3.8488*e*-04
Rateofconvergence					
10^{-L}	1.0107	1.0077	1.0043	1.0022	

After 10^{-2}	1.8542*e*-04	5.5542*e*-05	1.5119*e*-05	3.9402*e*-06	1.0055*e*-06
10^{-4}	2.2699*e*-04	6.6325*e*-05	1.7858*e*-05	4.6297*e*-06	1.1786*e*-06
10^{-6}	2.2742*e*-04	6.6437e-05	1.7886*e*-05	4.6369*e*-06	1.1804*e*-06
10^{-8}	2.2743*e*-04	6.6438*e*-05	1.7887*e*-05	4.6370*e*-06	1.1804*e*-06
10^{-10}	2.2743*e*-04	6.6438*e*-05	1.7887*e*-05	4.6370*e*-06	1.1804*e*-06
⋮	⋮	⋮	⋮	⋮	⋮
10^{-L}	2.2743*e*-04	6.6438*e*-05	1.7887*e*-05	4.6370*e*-06	1.1804*e*-06
Rateofconvergence					
10^{-L}	1.7753	1.8931	1.9476	1.9739	

We also assess the corresponding convergence rate..

$$R^{M,N} = \frac{\log E_{\varepsilon}^{M,N} - \log E_{\varepsilon}^{2M,2N}}{\log 2} \tag{8}$$

Example: Let A(x) = $(2 - x^2)$, B(x) = (x – 3), C(x) = −2, D (x) = −1, F(x, t) = $10t^2e^{-t}x(1 - x)$, where (x, t) ∈ (0, 1) × (0, 1], and with initial boundary condition,

$$\begin{aligned} u(x,t) &= 0, \quad \forall (x,t) \in \Omega_l = \{(x,t):\ -\gamma \le x \le 0, \text{and } 0 \le t \le 1\}, \\ u(x,t) &= 0, \quad \forall (x,t) \in \Omega_r = \{(x,t):\ 1 \le x \le 1+\mu, \text{and } 0 \le t \le 1\}, \\ u(x,t) &= 0, \quad \forall (x,t) \in \Omega_b = \{(x,t):\ 0 \le x \le 1, \text{and } 0 \le t \le 1\}. \end{aligned} \tag{9}$$

Example: Let A(x) = $(1 + x + x^2)$, B(x) = $(1 + x^2)$, C(x) = $-(0{:}25 + 0{:}5x^2)$, D(x) = −0:25, F(x, t) = sin (πx)(1 – x), where (x, t) ∈ (0, 1) × (0, 1], with initial and boundary condition,

$$\begin{aligned} u(x,t) &= 0, \quad \forall (x,t) \in \Omega_l = \{(x,t):\ -\gamma \le x \le 0, \text{and } 0 \le t \le 1\}, \\ u(x,t) &= 0, \quad \forall (x,t) \in \Omega_r = \{(x,t):\ 1 \le x \le 1+\mu, \text{and } 0 \le t \le 1\}, \\ u(x,t) &= 0, \quad \forall (x,t) \in \Omega_b = \{(x,t):\ 0 \le x \le 1, \text{and } 0 \le t \le 1\}. \end{aligned} \tag{10}$$

Example. Let A(x) = $(1 - x^2/2)$, B(x) = (x + 6), C(x) = −4, D(x) = −1, F(x, t) = x(1 – x), where (x, t) ∈ (0, 1) × (0, 3], with initial and boundary condition,

$$u(x,t)=0, \quad \forall(x,t)\in\Omega_l=\{(x,t): -\gamma\le x\le 0, \text{and } 0\le t\le 3\},$$
$$u(x,t)=0, \quad \forall(x,t)\in\Omega_r=\{(x,t): 1\le x\le 1+\mu, \text{and } 0\le t\le 3\},$$
$$u(x,t)=0, \quad \forall(x,t)\in\Omega_b=\{(x,t): 0\le x\le 1, \text{and } 0\le t\le 3\}. \qquad \textbf{(11)}$$

3. RESULTS & DISCUSSIONS

The approach for solving the spatially delayed singularly perturbed parabolic partial differential equation has been developed by us. Defining the model problem, decomposing it into three equations, approximating the time variable with implicit Euler's method, approximating the delay term with Taylor series expansion of order two, approximating the spatial variable with the central difference method, and locating the fitting factor are the fundamental mathematical procedures. The next step is to use the Richardson extrapolation method, which will increase the precision of the procedure.

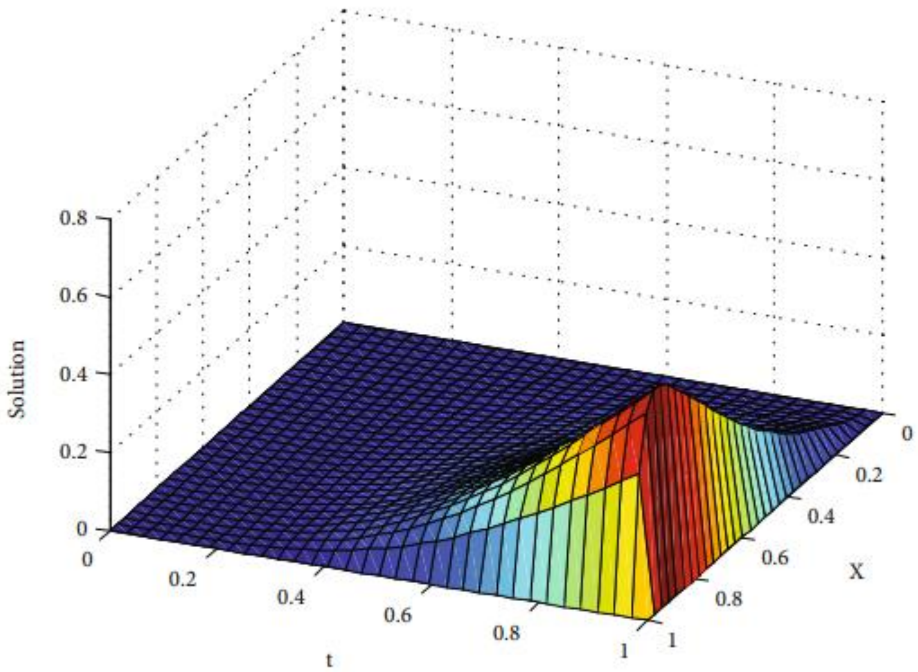

Figure 1: The physical properties of the solutions for Example 5.1 at m = n = 32, ε = 10^{-2}, γ = 0:5ε, and μ = 0:6ε

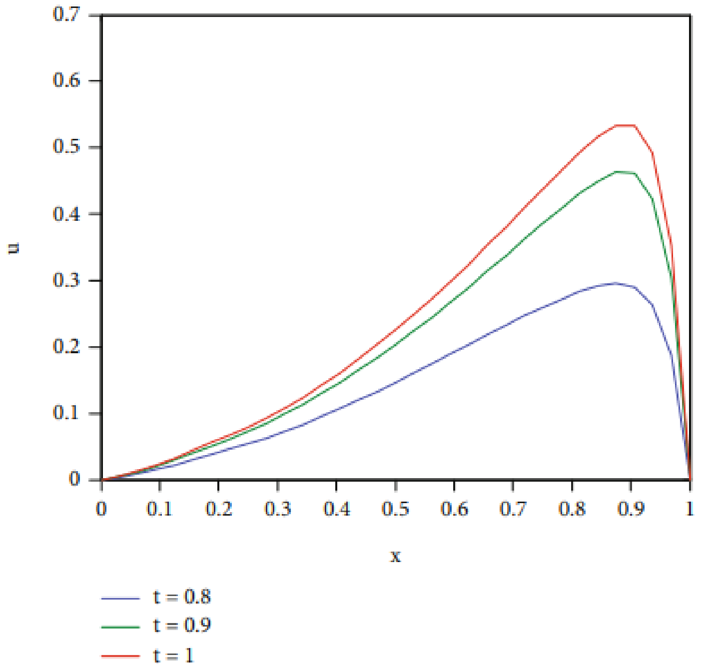

Figure 2: The physical behavior of the solutions for Example 5.1 at m = n = 32, $\varepsilon = 10^{-2}$, $\gamma = 0{:}5\varepsilon$, and $\mu = 0{:}6\varepsilon$ at various time level

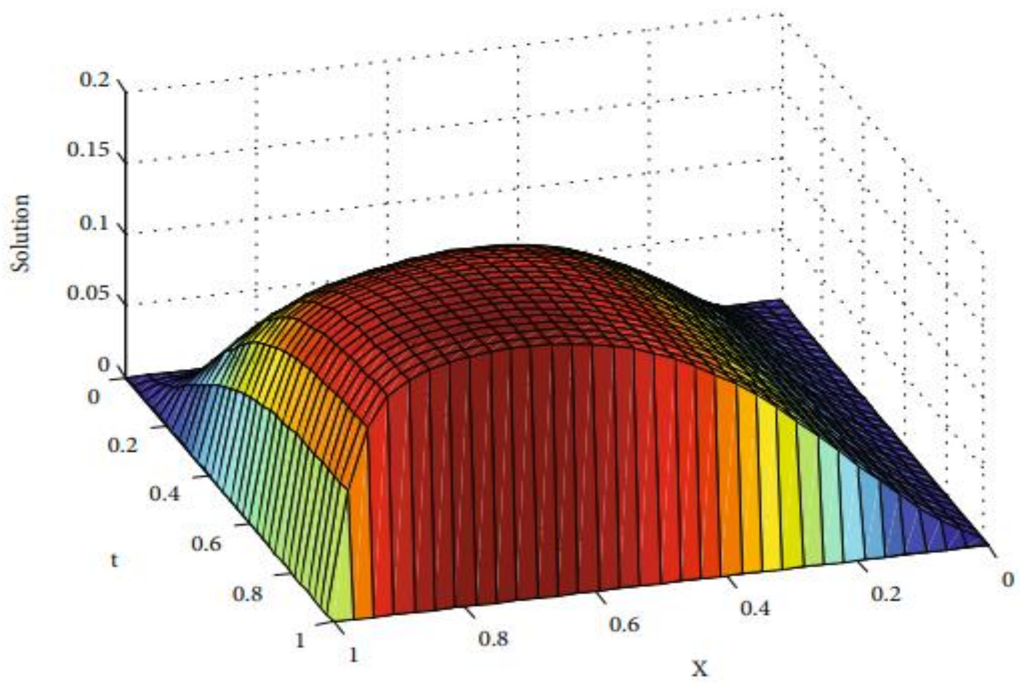

Figure 3: The physical behavior of the solutions for Example 5.2 at m = n = 64, $\varepsilon = 10^{-2}$, $\gamma = 0{:}5\varepsilon$, and $\mu = 0{:}6\varepsilon$

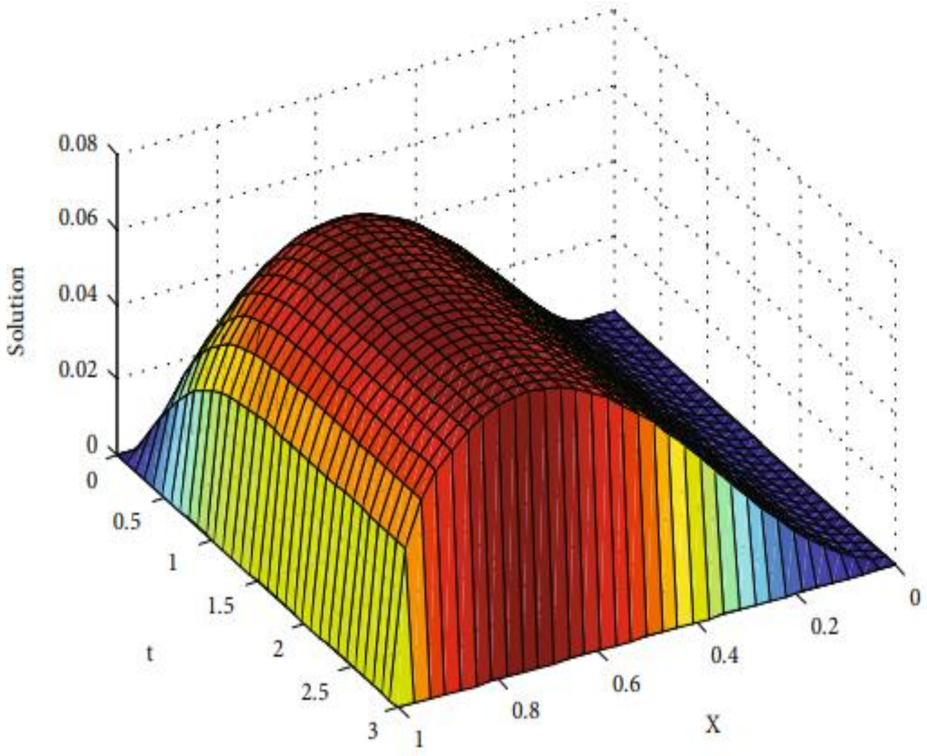

Figure 4: The physical behavior of the solutions for Example 5.3 at m = n = 32, $\varepsilon = 10^{-2}$, $\gamma = 0{:}1\varepsilon$, and $\mu = 0{:}2\varepsilon$:

4. CONCLUSION:

Three model cases are provided to demonstrate the suggested method's effectiveness. Table 1demonstrate the maximum error and rate of convergence for various values of ε, delay parameters, and mesh length. Figures 1–4 depict the solution's physicochemical properties. The proposed numerical technique was tested for stability, consistency, and uniform convergence. The current method is second-order convergent with regard to temporal and spatial variables, as shown in the results, and the rate of convergence is two and more accurate than some of the methods in the literature.

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